REGULARITY OF CONSERVATIVE INDUCTIVE LIMITS

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ABSTRACT. A sequentially complete inductive limit of Fréchet spaces is regular, see [3]. With a minor modification, this property can be extended to inductive limits of arbitrary locally convex spaces under an additional assumption of conservativeness.

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Throughout the paper $E_1 \subset E_2 \subset \dots$ is a sequence of Hausdorff locally convex spaces with continuous identity maps $\text{id} : E_n \rightarrow E_{n+1}$, $n \in N$. Their respective topologies are denoted by τ_n . The topology of their inductive limit $\text{ind } E_n$ is denoted by $\tau = \text{ind } \tau_n$.

We will use a result from [1, Cor. IV. 6.5]. It reads:

If F as well as all spaces E_n are Fréchet and $T : F \rightarrow \text{ind } E_n$ is a linear map with a closed graph, then there is $n \in N$ such that T is a continuous map of F into E_n .

According to [2, Sec. 5.2], the space $\text{ind } E_n$ is called α -regular, resp. regular, if every set bounded in $\text{ind } E_n$ is contained, resp. bounded, in some constituent space E_n . We will need a slightly modified notion of regularity.

DEFINITION 1. An inductive limit $\text{ind } E_n$ is quasi α -regular, resp. quasi regular, if every set bounded in $\text{ind } E_n$ is a subset of a τ -closure of a set contained, resp. bounded, in some constituent space E_n .

DEFINITION 2. An inductive limit $\text{ind } E_n$ is called conservative if for every linear subspace $F \subset \text{ind } E_n$, we have

$$\text{ind}(F \cap E_n, \tau_n) = (F, \text{ind } \tau_n). \quad (1)$$

LEMMA. Let a locally convex (Hausdorff) space E be sequentially complete, and B be a balanced, bounded, closed, and convex set in E . Then the linear span F of B , equipped with the topology generated by the Minkowski functional of B , is a Banach space and the identity map $\text{id} : F \rightarrow E$ is continuous.

PROOF. Clearly F is a normed space and $\text{id} : F \rightarrow E$ is continuous.

To prove the completeness of F , take a Cauchy sequence $\{x_n\}$ in F . Since $\text{id} : F \rightarrow E$ is continuous, $\{x_n\}$ is Cauchy in E . Hence it converges to some $x \in E$. The set $\bigcup \{x_n; n \in N\}$, which is bounded in F , is contained in some αB . Since the set αB is closed in E , we have $x \in \alpha B \subset F$.

For any 0-nbhd λB , $\lambda > 0$, in F , there exists $k \in N$ such that $m, n > k$ imply $x_n - x_m \in \lambda B$. If we let $m \rightarrow \infty$, we get $x_n - x \in \lambda B$ for $n > k$, i.e., $x_n \rightarrow x$ in F . $\square$

PROPOSITION 1. *Any sequentially complete $\text{ind } E_n$ is quasi α -regular.*

PROOF. Let a set A be bounded in $\text{ind } E_n$. Denote by B its balanced, convex, τ -closed hull, and by F the linear span of B with the same topology γ as in the Lemma. We know that F is a Banach space.

For any $n \in N$, denote by G_n the completion of the normed space $(F \cap E_n, \gamma)$. Then $G_n \subset F$ and F equals strict inductive limit $\text{ind } G_n$. Since B is bounded in F , it is bounded in $\text{ind } G_n$. Hence, by [1, Cor. IV. 6.5], B is bounded in some G_n .

Finally, $A \subset B$ and B is a γ -closure of a set $V = \bigcup \{E_n \cap \lambda B; 0 < \lambda < 1\}$ in $F \cap E_n$. Hence A is also a subset of the τ -closure of V in $\text{ind } E_n$. $\square$

PROPOSITION 2. *Let $\text{ind } E_n$ be sequentially complete and conservative. Then every set $A \subset E_1$, which is bounded in $\text{ind } E_n$ is also bounded in some constituent space E_n .*

PROOF. Take such A and assume that it is not bounded in any E_n . Then for any $n \in N$, there exists a balanced convex 0-nbhd U_n in E_n which does not absorb A . For any $m, n \in N$, choose $a_{m,n} \in A$ such that $a_{m,n} \notin mU_n$. Denote by B the τ -closure of the convex balanced hull of $\bigcup \{a_{m,n}; m, n \in N\}$ and by F the linear span of B . For any $m, n \in N$, there exists $f_{m,n} \in (\text{ind } E_n)'$, (the dual of $\text{ind } E_n$), such that $f_{m,n}(a_{m,n}) \neq 0$. Put $V_{m,n} = \{x \in F; |f_{m,n}(x)| \leq 1\}$ and denote by F_n the linear space F equipped with the topology generated by $\{U_m; m \geq n\} \cup \{V_{m,n}; m, n \in N\}$. Then each F_n is a metrizable Hausdorff locally convex space and its completion G_n is a Fréchet space.

Finally, let H be the space F equipped with the topology generated by the Minkowski functional of B . The set B is bounded in $\text{ind } E_n$, hence, by the Lemma, H is Banach space and the identity map $\text{id} : H \rightarrow \text{ind } E_n$ is continuous.

Since $\text{ind } E_n$ is conservative and $F \subset \text{ind } E_n$, we have

$$\text{ind}(F, \tau_n) = (F, \text{ind } \tau_n). \quad (2)$$

For any $n \in N$, the identity maps $(F, \tau_n) \rightarrow F_n \rightarrow G_n$ are continuous. Hence

$$\text{id} : \text{ind}(F, \tau_n) \rightarrow \text{ind } G_n \quad (3)$$

is continuous, too. Then, the continuity of $\text{id} : H \rightarrow \text{ind } E_n$ implies the continuity of $\text{id} : H \rightarrow (F, \text{ind } \tau_n)$. By (2) and (3), we finally get the continuity of $\text{id} : H \rightarrow \text{ind } G_n$.

By [1, Cor. IV. 6.5], there exists $n \in N$ such that $\text{id} : H \rightarrow G_n$ is continuous. Since the set B is bounded in H and contained in F_n , it is bounded in G_n , and also bounded in F_n . But then B , as well as its subset A , are absorbed by the 0-nbhd V_n in F_n , a contradiction. $\square$

By combining Propositions 1 and 2, we get

THEOREM. *Any sequentially complete conservative $\text{ind } E_n$ is quasi regular.*

COROLLARY. *If moreover each space E_n in the above Theorem is closed in $\text{ind } E_n$, then $\text{ind } E_n$ is regular.*

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