ON WEAK SOLUTION OF A HYPERBOLIC DIFFERENTIAL INCLUSION WITH NONMONOTONE DISCONTINUOUS NONLINEAR TERM

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ABSTRACT. In this paper, a hyperbolic differential inclusion with nonmonotone discontinuous and nonlinear term, which the generalized velocity acts as its variable, is studied and the existence and decay of its weak solution are obtained.

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1. Introduction. In the present paper, we investigate the initial boundary value problem of the following degenerate multi-valued hyperbolic differential inclusion:

$$\begin{aligned} \ddot{u}(t) + B(u)(t) + \varphi(\dot{u})(t) &\ni f(t), \quad \text{a.e. } t \in [0, T], \\ u(x, t) &= 0, \quad \text{a.e. } (x, t) \in \sum = \partial\Omega \times [0, T], \\ u(0) &= u_0, \quad \dot{u}(0) = u_1, \end{aligned} \tag{1.1}$$

where B is a linear and symmetric operator; φ is a discontinuous, nonmonotone, and nonlinear set-valued mapping.

Physical motivations for studying equation (1.1) come partly from problems of continuum mechanics, where nonmonotone, nonlinear, discontinuous, and multi-valued constitutive laws and boundary constraints lead to the above variational inequalities (differential inclusions). For example, when elastobody is constrained by boundary friction, (1.1) denotes its control equation; if we study viscoelastical body and the unilateral problem of plate, (1.1) is also their control equation, etc. [10, 8, 5].

When φ is a nonmonotone multi-valued mapping, generally, for such nonmonotone and discontinuous multi-valued systems, usual monotonicity methods are not valid [1, 6]. When φ degenerates into a class of single-valued mappings and satisfies appropriate conditions, inequation (1.1) become an equation. Equation (1.1) and some of its evolution equations with which it is associated have been investigated and applied intensively [7, 3, 2, 9, 11].

In this paper, we investigate the existence and decay of the weak solutions of the hyperbolic in equation (1.1), with φ and B satisfying adequate conditions under zero boundary conditions.

2. Preliminaries. Let Ω be a bounded open set of R^n with regular boundary Γ . Let T denote a positive real number, $Q = \Omega \times [0, T]$. Suppose that $b \in L_{\text{loc}}^\infty(R)$. For every

$\rho > 0$, set

$$\underline{b}_p(\xi) = \operatorname{ess\,inf}_{|\xi_1 - \xi| < p} b(\xi_1), \quad \overline{b}_p(\xi) = \operatorname{ess\,sup}_{|\xi_1 - \xi| < p} b(\xi_1), \quad (2.1)$$

and

$$\underline{b}(\xi) = \lim_{p \rightarrow 0^+} \underline{b}_p(\xi), \quad \overline{b}(\xi) = \lim_{p \rightarrow 0^+} \overline{b}_p(\xi), \quad \varphi(\xi) = [\underline{b}(\xi), \overline{b}(\xi)]. \quad (2.2)$$

Let $J(\xi) = \int_0^\xi b(t) dt$. Then $\partial^c J(\xi) \subseteq \varphi(\xi)$, where $\partial^c J(\xi)$ denotes the Clarke-subdifferential of J .

REMARK. If $b(\xi_\pm)$ exists for every $\xi \in R$, then $\varphi(\xi) = \partial^c J(\xi)$. Furthermore, if J is convex, $\varphi(\xi)$ is maximal monotone. If b is continuous at ξ , then $\varphi(\xi)$ is single-valued at ξ ([3]).

Let $V = H_0^1(\Omega)$, $\langle \cdot, \cdot \rangle$ denote the dual pair between $V = H_0^1(\Omega)$ and $V' = H^{-1}(\Omega)$, and $(\cdot, \cdot)$ the inner product of $L^2(\Omega)$ which is compatible with the dual pair. Let $|\cdot|_X$ denote the norm of an element x of a Banach space X .

Consider the following initial boundary value problem of a hyperbolic variational inequation (inclusion):

$$\begin{aligned} \dot{u}(t) + Bu(t) + g(t) &= f(t), \quad \text{a.e. } t \in [0, T], \\ g(x, t) &\in \varphi(\dot{u}(x, t)), \quad \text{a.e. } (x, t) \in Q_T = \Omega \times [0, T], \\ u(x, t) &= 0, \quad \text{a.e. } (x, t) \in \Sigma = \partial\Omega \times [0, T], \\ u(0) &= u_0, \quad \dot{u}(0) = u_1, \end{aligned} \quad (2.3)$$

where f , u_0 , and u_1 are given.

3. Existence of solution

THEOREM 1. Assume that $f \in L^2(0, T; L^2(\Omega))$, $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$. If

- (1) $\exists c > 0, |b(\xi)| \leq c(1 + |\xi|)$, a.e. $\xi \in R$,
- (2) $B : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ is linear, continuous, symmetric, and semicoercive, i.e., $\exists c_1 > 0, c_2 > 0, c_3 \geq 0$,

$$\begin{aligned} |Bv|_{H^{-1}(\Omega)} &\leq c_1 |v|_{H_0^1(\Omega)}, \\ \langle Bu, v \rangle &= \langle Bv, u \rangle \quad \forall u, v \in H_0^1(\Omega), \\ \langle Bv, v \rangle + c_3 |v|_{L^2(\Omega)}^2 &\geq c_2 |v|_{H_0^1(\Omega)}^2 \quad \forall v \in H_0^1(\Omega), \end{aligned} \quad (3.1)$$

then there exists a function u , defined in $\Omega \times [0, T]$, such that

$$\begin{aligned} u &\in L^\infty(0, T; H_0^1(\Omega)) \cap C([0, T]; L^2(\Omega)), \\ \dot{u} &\in L^\infty(0, T; L^2(\Omega)) \cap C([0, T]; H^{-1}(\Omega)), \\ \ddot{u} &\in L^2(0, T; H^{-1}(\Omega)), \end{aligned} \quad (3.2)$$

and

$$\begin{aligned} \ddot{u}(t) + Bu(t) + g(t) &= f(t) \quad \text{in } L^2(0, T; H^{-1}(\Omega)), \\ g(t) &\in \varphi(\dot{u}(x, t)), \quad \text{a.e. } (x, t) \in \Omega \times [0, T], \\ u(0) &= u_0, \quad \dot{u}(0) = u_1. \end{aligned} \quad (3.3)$$

PROOF. Let $\{e_n\}_{n=1}^\infty$ be a subset of $V = H_0^1(\Omega)$ satisfying $\overline{\text{span}\{e_n\}} = V$, $(e_i, e_j) = \delta_{ij}$. Moreover, let $x_n = \sum_1^n \omega_i^1 e_i \rightarrow u_0$ strongly in V , $y_n = \sum_1^n \omega_i^2 e_i \rightarrow u_1$ strongly in $L^2(\Omega)$.

Consider the following regularized equation of inequation (1.1)

$$\ddot{\xi}^n = N^n + h, \quad \xi^n|_{t=0} = \omega^{1n}, \quad \dot{\xi}^n|_{t=0} = \omega^{2n}, \quad (3.4)$$

where

$$\begin{aligned} \xi^n &= \{\xi_i^n\}_{1 \times n}, & \omega^{1n} &= \{\omega_i^1\}_{1 \times n}, & \omega^{2n} &= \{\omega_i^2\}_{1 \times n}, & h &= \{ \langle f, e_i \rangle \}_{1 \times n}, \\ N^n &= \{N_i^n\}_{1 \times n}, & N_i^n &= - \left\langle B \left(\sum_1^n \xi_j^n e_j \right), e_i \right\rangle - \left\langle b \left(\sum_1^n \dot{\xi}_j^n e_j \right), e_i \right\rangle, \end{aligned} \quad (3.5)$$

where “.” denotes time derivate.

Equation (3.4) is a set of second-order ordinary differential equation and its local solution ξ^n exists on $I_n = [0, T_n]$, $0 < T_n \leq T$.

Set $u_n(t) = \sum_1^n \xi_j^n e_j$ ($t \in I_n$). Equation (3.4) is equivalent to

$$\langle \ddot{u}_n, e_i \rangle = - \langle Bu_n, e_i \rangle - \langle b(\dot{u}_n), e_i \rangle + \langle f, e_i \rangle, \quad i = 1, 2, \dots, n. \quad (3.6)$$

Multiplying equation (3.6) by ξ_i^n , summing from $i = 1$ to $i = n$, and integrating over $[0, t]$ ($t \in I_n$), we get

$$\begin{aligned} & |\dot{u}_n(t)|_{L^2(\Omega)}^2 + \langle Bu_n(t), u_n(t) \rangle + 2 \int_0^t \langle b(\dot{u}_n), \dot{u}_n \rangle d\tau \\ &= 2 \int_0^t \langle f, \dot{u}_n \rangle d\tau + (y_n, y_n) + \langle Bx_n, x_n \rangle, \\ & |b(\dot{u}_n)|_{L^2(0,t;L^2(\Omega))}^2 = \int_0^t |b(\dot{u}_n)|_{L^2(\Omega)}^2 d\tau \leq c \int_0^t \int_\Omega (1 + |\dot{u}_n|)^2 dx d\tau \\ &\leq \frac{c}{2} \int_0^t (|\Omega| + |\dot{u}_n(t)|_{L^2(\Omega)}^2) d\tau \\ &\leq c_4 + \frac{c}{2} \int_0^t |\dot{u}_n(\tau)|_{L^2(\Omega)}^2 d\tau, \end{aligned} \quad (3.7)$$

where $|\Omega|$ denotes the Lebesgue measure of domain Ω .

$$\begin{aligned} \int_0^t \langle b(\dot{u}_n), \dot{u}_n \rangle d\tau &\leq |b(\dot{u}_n)|_{L^2(0,t;L^2(\Omega))} \cdot |\dot{u}_n|_{L^2(0,t;L^2(\Omega))} \\ &\leq \frac{1}{2} (|b(\dot{u}_n)|_{L^2(0,t;L^2(\Omega))}^2 + |\dot{u}_n|_{L^2(0,t;L^2(\Omega))}^2) \\ &\leq \frac{1}{2} \left\{ c_4 + \left(\frac{c}{2} + 1 \right) \int_0^t |\dot{u}_n(\tau)|_{L^2(\Omega)}^2 d\tau \right\}, \end{aligned} \quad (3.8)$$

$$\begin{aligned} \int_0^t \langle f, \dot{u}_n \rangle d\tau &\leq |f|_{L^2(0,T;L^2(\Omega))} \cdot |\dot{u}_n|_{L^2(0,t;L^2(\Omega))} \\ &\leq \frac{1}{2} (|f|_{L^2(0,T;L^2(\Omega))}^2 + |\dot{u}_n|_{L^2(0,t;L^2(\Omega))}^2). \end{aligned} \quad (3.9)$$

From (3.7), $\exists c_5 > 0$ such that

$$\begin{aligned} & |\dot{u}_n(t)|_{L^2(\Omega)}^2 + c_2 |u_n(t)|_{H_0^1(\Omega)}^2 \\ &\leq c_5 + c_3 |u_n(t)|_{L^2(\Omega)}^2 + \frac{1}{2} \left(\frac{c}{2} + 1 \right) \int_0^t |u_n(\tau)|_{L^2(\Omega)}^2 d\tau. \end{aligned} \quad (3.10)$$

We note that

$$\begin{aligned} u_n(t) &= u_n(0) + \int_0^t \dot{u}_n d\tau, \\ |u_n(t)|_{L^2(\Omega)} &\leq |u_n(0)|_{L^2(\Omega)} + \int_0^t |\dot{u}_n|_{L^2(\Omega)} d\tau. \end{aligned} \quad (3.11)$$

By Hölder inequality, $\exists c_6, c_7 > 0$ such that

$$|u_n(t)|_{L^2(\Omega)}^2 \leq c_6 + c_7 \int_0^t |\dot{u}_n(\tau)|_{L^2(\Omega)}^2 d\tau. \quad (3.12)$$

From (3.10) and (3.12), we obtain: $\exists c_8, c_9 > 0$ such that

$$|\dot{u}_n(t)|_{L^2(\Omega)}^2 + c_2 |u_n(t)|_{H_0^1(\Omega)}^2 \leq c_8 + c_9 \int_0^t |\dot{u}_n(\tau)|_{L^2(\Omega)}^2 d\tau, \quad (t \in I_n). \quad (3.13)$$

Hence,

$$|\dot{u}_n(t)|_{L^2(\Omega)}^2 \leq c_8 + c_9 \int_0^t |\dot{u}_n(\tau)|_{L^2(\Omega)}^2 d\tau, \quad (t \in I_n). \quad (3.14)$$

By Gronwall's inequality, we have

$$|\dot{u}_n(t)|_{L^2(\Omega)}^2 \leq c_8 \exp(c_9 t), \quad (t \in I_n). \quad (3.15)$$

Therefore, from (3.12), (3.15), and (3.16), there exists $c_{10} > 0$ such that

$$|\dot{u}_n(t)|_{L^2(\Omega)} \leq C_{10}, \quad |u_n(t)|_{L^2(\Omega)} \leq C_{10}, \quad |u_n(t)|_{H_0^1(\Omega)} \leq C_{10}, \quad (t \in I_n), \quad (3.16)$$

where $c_4, c_5, c_6, c_7, c_8, c_9, c_{10}$ are positive constants independent of n and T_n . The estimate (3.16) implies that we can prolongate the solution of equation (3.4) to the interval $[0, T]$, i.e., $I_n = [0, T] \ (\forall n)$.

From (3.6), we see that, for every $\eta \in \text{span}\{e_1, e_2, \dots, e_n\}$,

$$\begin{aligned} |\langle \ddot{u}_n, \eta \rangle| &\leq |f(t)|_{L^2(\Omega)} \cdot |\eta|_{L^2(\Omega)} + |b(\dot{u}_n)|_{L^2(\Omega)} \cdot |\eta|_{L^2(\Omega)} \\ &\quad + |B| \cdot |u_n|_{H_0^1(\Omega)} \cdot |\eta|_{H_0^1(\Omega)}, \end{aligned} \quad (3.17)$$

where $|B|$ is the norm of the linear continuous operator B .

$$\begin{aligned} |\ddot{u}_n(t)|_{H^{-1}(\Omega)} &= \sup_{|\eta|_V=1} |\langle \ddot{u}_n(t), \eta \rangle| = \sup_{\substack{\eta \in \text{span}\{e_1, \dots, e_n\} \\ |\eta|_V=1}} |\langle \ddot{u}_n(t), \eta \rangle| \\ &\leq c_{11} \left(|f(t)|_{L^2(\Omega)} + |b(\dot{u}_n)|_{L^2(\Omega)} + |B| \cdot |u_n(t)|_{H_0^1(\Omega)} \right), \end{aligned} \quad (3.18)$$

where c_{11} is the imbedding constant which $H_0^1(\Omega)$ imbeds in $L^2(\Omega)$.

$$|b(\dot{u}_n)(t)|_{L^2(\Omega)}^2 \leq c \int_{\Omega} (1 + |\dot{u}_n(t)|)^2 dx \leq \frac{c}{2} (|\Omega| + |\dot{u}_n(t)|_{L^2(\Omega)}^2). \quad (3.19)$$

This shows that $\{b(\dot{u}_n)\}$ is also a bounded subset of $L^\infty(0, T; L^2(\Omega))$. Hence, (3.18) implies that $\{\ddot{u}_n\}$ is a bounded subset of $L^2(0, T; H^{-1}(\Omega))$.

Therefore, there exists a subsequence of $\{u_n\}$ (still denoted by itself) and a function u such that $u \in L^\infty(0, T; H_0^1(\Omega))$, $\dot{u} \in L^\infty(0, T; L^2(\Omega))$, $\ddot{u} \in L^2(0, T; H^{-1}(\Omega))$ satisfying

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{weakly-star in } L^\infty(0, T; H_0^1(\Omega)), \\ \dot{u}_n &\rightharpoonup \dot{u} \quad \text{weakly-star in } L^\infty(0, T; L^2(\Omega)), \\ \ddot{u}_n &\rightharpoonup \ddot{u} \quad \text{weakly in } L^2(0, T; H^{-1}(\Omega)), \\ b(\dot{u}_n) &\rightharpoonup g \quad \text{weakly-star in } L^\infty(0, T; L^2(\Omega)). \end{aligned} \quad (3.20)$$

Furthermore, $\dot{u}_n(t, x) \rightarrow \dot{u}(t, x)$, a.e. $(t, x) \in [0, T] \times \Omega$.

It is well known that the space $W(V)$, defined by $W(V) = \{u \in L^2(0, T; V), \dot{u} \in L^2(0, T; V')\}$ with the norm $\|u\|_W = \|u\|_{L^2(0, T; V)} + \|\dot{u}\|_{L^2(0, T; V')}$, is continuously imbedded in $C([0, T]; L^2(\Omega))$. It is obvious that $u \in C(0, T; L^2(\Omega))$, $\dot{u} \in C(0, T; H^{-1}(\Omega))$. Hence, $u(0)$, $\dot{u}(0)$ make sense.

For $\lambda \in L^2(0, T)$, by (3.6), we have

$$\begin{aligned} \int_0^T \langle \ddot{u}_n, \lambda e_i \rangle dt &= - \int_0^T \langle B(u_n), \lambda e_i \rangle dt - \int_0^T \langle b(\dot{u}_n), \lambda e_i \rangle dt \\ &\quad + \int_0^T \langle f(t), \lambda e_i \rangle dt, \quad i = 1, 2, \dots, n. \end{aligned} \quad (3.21)$$

For every given positive integer i , let $n \rightarrow \infty$ in (3.21). Then, it follows that

$$\begin{aligned} \int_0^T \langle \ddot{u}, \lambda e_i \rangle dt &= - \int_0^T \langle B(u), \lambda e_i \rangle dt - \int_0^T \langle g, \lambda e_i \rangle dt \\ &\quad + \int_0^T \langle f(t), \lambda e_i \rangle dt, \quad i = 1, 2, \dots \end{aligned} \quad (3.22)$$

Therefore,

$$\ddot{u}(t) + B(u) + g(t) = f(t) \quad \text{in } L^2(0, T; H^{-1}(\Omega)). \quad (3.23)$$

In the following, we show that

$$g(x, t) \in \varphi(\dot{u}(x, t)), \quad \text{a.e. } (x, t) \in Q_T = \Omega \times [0, T]. \quad (3.24)$$

Since $\dot{u}_n(x, t) \rightarrow \dot{u}(x, t)$, a.e. $(x, t) \in Q_T$, by Eropob's theorem [12], for every $\delta > 0$, there exists a subset $Q_\delta \subseteq Q_T = \Omega \times [0, T]$, $|Q_\delta| \leq \delta$,

$$\dot{u}_n(x, t) \rightarrow \dot{u}(x, t) \quad \text{uniformly in } Q_T / Q_\delta. \quad (3.25)$$

That is, for every $\varepsilon > 0$, there exists a positive integer $\bar{N}$, when $n \geq \bar{N}$,

$$|\dot{u}_n(x, t) - \dot{u}(x, t)| \leq \varepsilon \quad \forall (x, t) \in Q_T / Q_\delta. \quad (3.26)$$

It is obvious that

$$\underline{b}_\varepsilon(\dot{u}(x, t)) \leq b(\dot{u}_n(x, t)) \leq \bar{b}_\varepsilon(\dot{u}(x, t)) \quad \forall (x, t) \in Q_T / Q_\delta. \quad (3.27)$$

For every $\mu \in L^1(0, T; L^2(\Omega))$, $\mu \geq 0$

$$\begin{aligned} \int_{Q_T \setminus Q_\delta} g(x, t) \mu(x, t) dx dt &= \lim_{n \rightarrow \infty} \int_{Q_T \setminus Q_\delta} b(\dot{u}_n(x, t)) \mu(x, t) dx dt \\ &\leq \int_{Q_T \setminus Q_\delta} \bar{b}_\varepsilon(\dot{u}(x, t)) \mu(x, t) dx dt, \end{aligned} \quad (3.28)$$

$$\begin{aligned} \int_{Q_T \setminus Q_\delta} g(x, t) \mu(x, t) dx dt &\leq \limsup_{\varepsilon \rightarrow 0^+} \int_{Q_T \setminus Q_\delta} \bar{b}_\varepsilon(\dot{u}(x, t)) \mu(x, t) dx dt \\ &\leq \int_{Q_T \setminus Q_\delta} \bar{b}(\dot{u}(x, t)) \mu(x, t) dx dt. \end{aligned} \quad (3.29)$$

Analogously, we can obtain

$$\int_{Q_T \setminus Q_\delta} g(x, t) \mu(x, t) dx dt \geq \int_{Q_T \setminus Q_\delta} \underline{b}(\dot{u}(x, t)) \mu(x, t) dx dt. \quad (3.30)$$

Hence,

$$g(x, t) \in \varphi(\dot{u}(x, t)), \quad \text{a.e. } (x, t) \in Q_T / Q_\delta. \quad (3.31)$$

Letting $\delta \rightarrow 0^+$, we get

$$g(x, t) \in \varphi(\dot{u}(x, t)), \quad \text{a.e. } (x, t) \in Q_T = \Omega \times [0, T]. \quad (3.32)$$

Let $\lambda \in C^1[0, T]$, $\lambda(T) = 0$. Integrating by parts the left-hand sides of equations (3.21) and (3.22) gives

$$\begin{aligned} -\langle \dot{u}_n(0), \lambda(0)e_i \rangle - \int_0^T \langle \dot{u}_n, \dot{\lambda}e_i \rangle dt &= \text{the right of (3.21),} \\ -\langle \dot{u}(0), \lambda(0)e_i \rangle - \int_0^T \langle \dot{u}, \dot{\lambda}e_i \rangle dt &= \text{the right of (3.22).} \end{aligned} \quad (3.33)$$

Making a comparison between the two equations of (3.33), we get

$$\lim_{n \rightarrow \infty} \langle \dot{u}_n(0) - \dot{u}(0), e_i \rangle = 0, \quad i = 1, 2, \dots \quad (3.34)$$

Therefore,

$$\dot{u}_n(0) \rightarrow \dot{u}(0) \text{ weakly in } H^{-1}(\Omega). \quad (3.35)$$

The uniqueness of the limit implies that $\dot{u}(0) = u_1$ (in $H^{-1}(\Omega)$).

Let $\lambda \in C^2[0, T]$, $\lambda(T) = 0$, $\dot{\lambda}(T) = 0$. Analogously, integrating by parts the left-hand sides of equations (3.33), and making a comparison with the obtained results again gives: $u(0) = u_0$ (in $L^2(\Omega)$). This completes the proof. $\square$

THEOREM 2. Let $f \in L^2(0, T; L^2(\Omega))$, $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$. Assume that b satisfies (1)' $b(\xi)\xi \geq -\delta$ for almost everywhere $\xi \in \mathbb{R}$, and $\exists \bar{c} > 0$, $|b(\xi)| \leq \bar{c}(1 + |\xi|^p)$, a.e. $\xi \in \mathbb{R}$, if $n > 2$, $0 < p \leq (2n)/(n-2)$; if $n \leq 2$, $0 \leq p < \infty$, and condition (2) of Theorem 1 is valid. Then there exists a function v , defined in $\Omega \times [0, T]$, satisfying

$$v \in L^\infty(0, T; H_0^1(\Omega)), \quad \dot{v} \in L^\infty(0, T; L^2(\Omega)), \quad (3.36)$$

and

$$\begin{aligned} \ddot{v} + B(v) + \bar{g}(t) &= f(t) \quad \text{in } L^1(0, T; H^{-1}(\Omega) + L^1(\Omega)), \\ \bar{g}(x, t) &\in \varphi(\dot{v}(x, t)), \quad \text{a.e. } (x, t) \in Q_T = \Omega \times [0, T], \\ v(0) &= u_0, \quad \dot{v}(0) = u_1. \end{aligned} \quad (3.37)$$

PROOF. Analogously to Theorem 1, we still may get (3.7), where $\{e_n\}_{n=1}^\infty$ is a basis of $H_0^1(\Omega) \cap L^\infty(\Omega)$ satisfying $(e_i, e_j) = \delta_{ij}$. Under assumption (1)', $\int_0^t \langle b(\dot{u}_n), \dot{u}_n \rangle d\tau \geq -\delta$. From (3.7), we have

$$|\dot{u}_n(t)|_{L^2(\Omega)}^2 + c_2 |u_n(t)|_{H_0^1(\Omega)}^2 \leq c_4 + c_3 |u_n(t)|_{L^2(\Omega)}^2 + 2 \int_0^t \langle f, \dot{u}_n \rangle d\tau. \quad (3.38)$$

It is easy to see that equations (3.12), (3.13), (3.15), and (3.16) are still true and the solution of equation (3.4) may still be extended to the interval $[0, T]$.

By Sobolev Imbedding Theorem, we have, for a.e. $t \in [0, T]$, if $n > 2$, then

$$H_0^1(\Omega) \subset L^{p^*}(\Omega) \subset L^p(\Omega), \quad p^* = \frac{2n}{n-2}, \quad (3.39)$$

and if $n = 2$, then

$$H_0^1(\Omega) \subset L^q(\Omega) \quad \forall 1 \leq q \leq \infty, \quad (3.40)$$

so

$$|u_n(t)|_{L^p(\Omega)} \leq c_{11} |u_n(t)|_{H_0^1(\Omega)} \leq c_{11} c_9; \quad (3.41)$$

if $n = 1$, then

$$H_0^1(\Omega) \subset C(\bar{\Omega}), \text{ and ditto, } |u_n(t)|_{C(\bar{\Omega})} = \max_{x \in \bar{\Omega}} |u_n(x, t)| \leq c_{11} c_9, \quad (3.42)$$

where $\bar{\Omega}$ denotes the Closure of Ω and c_{11} is the imbedding constant which $H_0^1(\Omega)$ imbeds in $L^p(\Omega)$ or $C(\bar{\Omega})$. Everyway, we always have that $b(\dot{u}_n) \in L^\infty(0, T; L^1(\Omega))$ and $\{b(\dot{u}_n)\}$ is a bounded subset of $L^\infty(0, T; L^1(\Omega))$.

Therefore, there exists a subsequence of $\{u_n\}$, still denoted by itself, and a function v , such that $v \in L^\infty(0, T; H_0^1(\Omega))$, $\dot{v} \in L^\infty(0, T; L^2(\Omega))$, satisfying

$$\begin{aligned} u_n &\rightharpoonup v \quad \text{weakly-star in } L^\infty(0, T; H_0^1(\Omega)), \\ \dot{u}_n &\rightharpoonup \dot{v} \quad \text{weakly-star in } L^\infty(0, T; L^2(\Omega)), \\ b(\dot{u}_n) &\rightharpoonup \bar{g} \quad \text{weakly-star in } L^\infty(0, T; L^1(\Omega)). \end{aligned} \quad (3.43)$$

Since the dual of the space $H_0^1(\Omega) \cap L^\infty(\Omega)$ is the space $H^{-1}(\Omega) + L^1(\Omega)$, by (3.6), it is easy to obtain

$$\dot{v}(t) + B(v) + \bar{g}(t) = f(t) \quad \text{in } L^1(0, T; H^{-1}(\Omega) + L^1(\Omega)). \quad (3.44)$$

The rest is analogous to that of Theorem 1.

This completes the proof. $\square$

4. Decay of solution

THEOREM 3. Let $T = +\infty$, $f \equiv 0$. Suppose that $\langle Bw, w \rangle \geq 0, \forall w \in H_0^1(\Omega)$. If $(b(w), w) \geq \mu_0 |w|_{L^2(\Omega)}^2$, then, under the conditions of Theorem 2, the solution in Theorem 2, obtained from the regularized equation (3.4), satisfies

$$|\dot{u}(t)|_{L^2(\Omega)}^2 \leq \mu_1 \exp(-\mu_2 t), \quad \text{a.e. } t \geq 0, \quad (4.1)$$

where μ_0, μ_1 , and μ_2 are positive constants.

PROOF. Let u_n be a solution of (3.4), i.e., u_n satisfies (3.6) and (3.7). Since $\langle b(w), w \rangle \geq \mu_0 \|w\|_{L^2(\Omega)}^2$, by (3.7), we have

$$\|\dot{u}_n(t)\|_{L^2(\Omega)}^2 + \langle Bu_n(t), u_n(t) \rangle \leq c_{12} - 2\mu_0 \int_0^t \|\dot{u}_n(\tau)\|_{L^2(\Omega)}^2 d\tau, \quad t \in [0, +\infty), \quad (4.2)$$

where c_{12} is a positive constant independent of n .

If $\langle Bw, w \rangle \geq 0, \forall w \in H_0^1(\Omega)$ and $\langle Bu_n(t), u_n(t) \rangle \geq 0$, then, by Gronwall inequality,

$$\|\dot{u}_n(t)\|_{L^2(\Omega)}^2 \leq c_{12} \exp(-2\mu_0 t), \quad \text{a.e. } t \geq 0. \quad (4.3)$$

Since

$$\|\dot{u}_n(t)\|_{L^2(\Omega)} \leq c_9, \quad \dot{u}_n \rightharpoonup \dot{u} \text{ weakly-star in } L^\infty(0, \infty; L^2(\Omega)), \quad (4.4)$$

it is easy to obtain that $\dot{u}_n(t) \rightharpoonup \dot{u}(t)$ weakly in $L^2(\Omega)$ for almost everywhere $t \geq 0$. But $L^2(\Omega)$ is a real Hilbert space, hence, $\|\dot{u}(t)\|_{L^2(\Omega)} \leq \varliminf_{n \rightarrow \infty} \|\dot{u}_n(t)\|_{L^2(\Omega)}$, a.e. $t \geq 0$ (see [4]). Finally, we get

$$\|\dot{u}(t)\|_{L^2(\Omega)}^2 \leq c_{12} \exp(-2\delta_0 t), \quad (\text{a.e. } t \geq 0). \quad (4.5)$$

□

REMARK 1. If $Bu = -\Delta u$, $\varphi(u) = |u|^p u$ then (1.1) is the equation which was ever considered by J. L. Lions [6]. J. L. Lions ever obtained the existence and uniqueness. But at this case, the result of decay of solution is true since the conditions of Theorem 3 is satisfied.

REMARK 2. When $Bu = -\Delta u$ and φ denotes the friction potential, equation (1.1) was considered by P. D. Panagiotopoulos under stronger conditions [8].

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