STABILITY FOR RANDOMLY WEIGHTED SUMS OF RANDOM ELEMENTS

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ABSTRACT. Let $\{X_n : n = 1, 2, 3, \dots\}$ be a sequence of i.i.d. random elements taking values in a separable Banach space of type p and let $\{A_{n,i} : i = 1, 2, 3, \dots; n = 1, 2, 3, \dots\}$ be an array of random variables. In this paper, under various assumptions of $\{A_{n,i}\}$, the necessary and sufficient conditions for $\sum_{i=1}^{\infty} A_{n,i} X_i \rightarrow 0$ a.s. are obtained. Also, the necessity of the assumptions of $\{A_{n,i}\}$ is discussed.

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1. Introduction. Let $\{X_n : n = 1, 2, 3, \dots\}$ be a sequence of independent identically distributed (i.i.d.) random variables and let $\{a_{n,i} : i = 1, 2, \dots, n; n = 1, 2, 3, \dots\}$ be a triangular array of constants. Many papers were devoted to extending various types of convergence modes to weighted sums $W_n = \sum_{i=1}^n a_{n,i} X_i$ in the literature. However, we are only interested in the work of almost sure convergence. The sequence $\{\sum_{i=1}^n a_{n,i}\}$ converging to 0 at a certain rate as $n \rightarrow \infty$ is a traditional assumption. For example, under the assumption $\sum_{i=1}^n a_{n,i}^2 = O(n^{-2/r})$, $W_n \rightarrow 0$ a.s. if $E|X_1|^r < \infty$ and $EX_1 = 0$ (See Chow and Lia [3] and Choi and Sung [2]). On the other hand, Padgett and Taylor [4] extended the usual convergence theorems to weighted sums of random elements in a separable Banach space. It would be interesting to extend the results with random weights.

Let $\{A_{n,i} : i = 1, 2, \dots, n; n = 1, 2, 3, \dots\}$ be a triangular array of random variables such that $\sum_{i=1}^n A_{n,i}^2 = O(n^{-2/r})$ a.s., Ahmad [1] obtained $W_n = \sum_{i=1}^n A_{n,i} X_i \rightarrow 0$ a.s. if $E\|X_1\|^r < \infty$ and $EX_1 = 0$. We note that, for the Marcinkiewicz-Zygmund law of large numbers, we take the uniform weight $a_{n,i} = n^{-1/r}$ but the condition $\sum_{i=1}^n a_{n,i}^2 = O(n^{-2/r})$ cannot be satisfied. The purpose of this paper is to extend the randomly weighted sums of a triangular array of random variables to that of an infinite array of random elements such that the Marcinkiewicz-Zygmund law of large numbers can be obtained as a corollary.

In Section 2, we establish the Marcinkiewicz-Zygmund law of large numbers in a separable Banach space of Type p . In Section 3, we consider an infinite array of random variables $\{A_{n,i} : n, i = 1, 2, 3, \dots\}$ as the weight under various assumptions of $\{A_{n,i}\}$, we obtain that $W_n = \sum_{i=1}^{\infty} A_{n,i} X_i \rightarrow 0$ a.s. if and only if $EX_1 = 0$ (when it exists) and $E\|X_1\|^r < \infty$.

2. The Marcinkiewicz law in a space of type p . Let (Ω, F, P) be a probability space and $\mathbf{B}$ be a real separable Banach space with norm $\|\bullet\|$. A random element is defined to be an F -measurable mapping of Ω into $\mathbf{B}$ with the Borel σ -field. The concept of independent random elements is a direct extension of the concept of independent random variables. A detailed account of basic properties of random elements in real Banach spaces can be found in Taylor [6].

In this section, we prove the Marcinkiewicz-Zygmund law of large numbers in a space of type p . First, we introduce a space of type p .

DEFINITION 1. Let $1 \leq p \leq 2$ and $\{r_i : i = 1, 2, 3, \dots\}$ be a sequence of independent random variables with $\Pr(r_i = \pm 1) = 1/2$. A separable Banach space $\mathbf{B}$ is said to be of type p if there exists a constant C such that

$$E \left\| \sum_{i=1}^n r_i x_i \right\| \leq C \left(\sum_{i=1}^n \|x_i\|^p \right)^{1/p} \quad (2.1)$$

for every $n \in N$ and all $x_1, \dots, x_n \in \mathbf{B}$.

Woyczyński [7] proved the equivalent condition of a space of type p .

LEMMA 1 (Woyczyński [7]). *Let $1 \leq p \leq 2$ and $q \geq 1$. The following properties of $\mathbf{B}$ are equivalent :*

- (i) *The separable Banach space $\mathbf{B}$ is of type p .*
- (ii) *There exists C such that, for every $n \in N$ and for any sequence $\{X_i : i = 1, 2, \dots, n\}$ of independent random elements in $\mathbf{B}$ with $EX_i = 0$, $i = 1, 2, \dots, n$,*

$$E \left(\left\| \sum_{i=1}^n X_i \right\|^q \right) \leq CE \left(\left(\sum_{i=1}^n \|X_i\|^p \right)^{q/p} \right). \quad (2.2)$$

Using Lemma 1, some elementary properties of spaces of type p can be easily proved. Every separable Hilbert space and finite-dimensional Banach space are of type 2. Every separable Banach space is at least of type 1, and the ℓ^p and L^p are of type $\min\{2, p\}$ for $p \geq 1$. If $\mathbf{B}$ is a space of type p and $1 \leq q \leq p$, then $\mathbf{B}$ is a space of type q . Before considering the Marcinkiewicz-Zygmund law of large numbers in a space of type p , we need the following definition and lemmas.

DEFINITION 2. Let $\mathbf{B}$ be a separable Banach space, $\mathbf{B}^*$ the dual space of $\mathbf{B}$, and $\mathbf{B}'$ the unit ball in $\mathbf{B}^*$. X is a random element in $\mathbf{B}$. The directionally maximum median of X is defined by

$$\rho(X) \equiv \sup_{f \in \mathbf{B}'} |\mu(f(X))|, \quad (2.3)$$

where $\mu(Y)$ denotes the minimum median in absolute value of the random variable Y .

LEMMA 2 (Sakhanenko [5]). *Let $X_1, \dots, X_n$ be independent random elements in $\mathbf{B}$ and $S_k = \sum_{i=1}^k X_i$, then, for every $t > 0$,*

$$\Pr \left(\max_{1 \leq k \leq n} \|S_k\| > t \right) \leq 2 \Pr \left(\|S_n\| > t - \max_{1 \leq k \leq n} \rho(S_n - S_k) \right). \quad (2.4)$$

LEMMA 3. *Let $\{X_n : n = 1, 2, 3, \dots\}$ be a sequence of independent random elements in a separable Banach space. If $S_n = \sum_{i=1}^n X_i$ converges to a random element S in probability, then S_n converges to S a.s.*

PROOF. Since S_n converges to S in probability, take ϵ such that $0 < \epsilon < 1/2$, then there exists an integer n_0 such that if $m > n \geq n_0$,

$$\Pr\left(\|S_m - S\| > \frac{\epsilon}{2}\right) < \frac{\epsilon}{2} \quad \text{and} \quad \Pr\left(\|S_n - S\| > \frac{\epsilon}{2}\right) < \frac{\epsilon}{2}. \quad (2.5)$$

So,

$$\Pr\left(\|S_m - S_n\| > \epsilon\right) < \Pr\left(\|S_m - S\| > \frac{\epsilon}{2}\right) + \Pr\left(\|S_n - S\| > \frac{\epsilon}{2}\right) < \epsilon < \frac{1}{2}. \quad (2.6)$$

We have $\mu(\|S_m - S_n\|) < \epsilon$ for any $m > n \leq n_0$, where $\mu(Y)$ is minimum median in absolute value of the random variable Y .

By Lemma 2, if $m_1 > n > n_0$,

$$\begin{aligned} \Pr\left(\max_{n < m < m_1} \|S_m - S_n\| > 2\epsilon\right) &\leq 2\Pr\left(\|S_{m_1} - S_n\| > 2\epsilon - \max_{n < m < m_1} \rho(S_{m_1} - S_m)\right) \\ &\leq 2\Pr\left(\|S_{m_1} - S_n\| > 2\epsilon - \max_{n < m < m_1} \mu(\|S_{m_1} - S_m\|)\right) \\ &\leq 2\Pr\left(\|S_{m_1} - S_n\| > \epsilon\right) < 2\epsilon. \end{aligned} \quad (2.7)$$

Let $m_1 \rightarrow \infty$, then if $m > n > n_0$, we have $\Pr(\max_{n < m} \|S_m - S_n\| > 2\epsilon) < 2\epsilon$.

We obtain S_n converges to some random element a.s., and S_n converges to S in probability. Hence, S_n converges to S a.s. $\square$

Now, we prove the Marcinkiewicz-Zygmund law of large numbers in a space of type p .

THEOREM 1. *Let $\mathbf{B}$ be a separable Banach space of type P and $\{X_n : n = 1, 2, 3, \dots\}$ be a sequence of independent and identically distributed random elements in $\mathbf{B}$. Then, for any $0 < r < p$,*

$$\frac{S_n - nc}{n^{1/r}} \rightarrow 0 \quad \text{a.s.} \quad (2.8)$$

for some constant c if and only if $E\|X_1\|^r < \infty$.

Moreover, if $r \geq 1$, $c = EX_1$ and $0 < r < 1$, c is arbitrary.

PROOF.

NECESSARY PART. Since (2.8) holds,

$$\frac{X_n}{n^{1/r}} = \frac{S_n - nc}{n^{1/r}} - \left(\frac{n-1}{n}\right)^{1/r} \frac{S_{n-1} - nc}{(n-1)^{1/r}} \rightarrow 0 \quad \text{a.s.}, \quad (2.9)$$

whence, by the Borel-Cantelli lemma, $\sum_{n=1}^{\infty} \Pr(\|X_1\| > n^{1/r}) < \infty$. Thus, $E\|X_1\|^r < \infty$.

SUFFICIENT PART. Since $E\|X_1\|^r < \infty$, define $Y_n = n^{-1/r} X_n I(\|X_n\| \leq n^{1/r})$ and $A_j = \{(j-1)^{1/r} < \|X_1\| \leq j^{1/r}\}$. Choose a positive number α such that $r < \alpha \leq p$ and $\alpha \geq 1$. We have

$$\sum_{n=1}^{\infty} E\|Y_n\|^\alpha = \sum_{n=1}^{\infty} \sum_{j=1}^n n^{-\alpha/r} \int_{A_j} \|X_1\|^\alpha dp = \sum_{j=1}^{\infty} \sum_{n=j}^{\infty} n^{-\alpha/r} \int_{A_j} \|X_1\|^\alpha dp$$

$$\begin{aligned}
&\leq c_1 \sum_{j=1}^{\infty} j^{1-\alpha/r} \int_{A_j} \|X_1\|^\alpha dp \leq c_1 \sum_{j=1}^{\infty} \int_{A_j} \|X_1\|^r dp \\
&= c_1 E \|X_1\|^r < \infty, \quad \text{for some constant } c_1.
\end{aligned} \tag{2.10}$$

From Lemma 1,

$$\begin{aligned}
E \left\| \sum_{i=n}^m (Y_i - EY_i) \right\|^\alpha &\leq c_2 E \left(\sum_{i=n}^m \|Y_i - EY_i\|^p \right)^{\alpha/p} \\
&\leq c_2 \sum_{i=n}^m E \|Y_i - EY_i\|^\alpha \leq 2^\alpha c_2 \sum_{i=n}^m E \|Y_i\|^\alpha.
\end{aligned} \tag{2.11}$$

We have $\sum_{i=1}^n (Y_i - EY_i)$ converges to some random element Y_0 in L^α . Therefore, $\sum_{i=1}^n (Y_i - EY_i) \rightarrow Y_0$ in probability. By Lemma 3, $\sum_{i=1}^n (Y_i - EY_i) \rightarrow Y_0$ a.s. Since

$$\sum_{n=1}^{\infty} \Pr \left(\frac{X_n}{n^{1/r}} \neq Y_n \right) = \sum_{n=1}^{\infty} \Pr (\|X_1\| > n^{1/r}) \leq E \|X_1\|^r < \infty, \tag{2.12}$$

so,

$$\sum_{n=1}^{\infty} \frac{X_n - E(X_n I(\|X_n\| \leq n^{1/r}))}{n^{1/r}} = \sum_{n=1}^{\infty} \left(\frac{X_n}{n^{1/r}} - EY_n \right) \quad \text{converges a.s.} \tag{2.13}$$

If $0 < r < 1$, we can choose $\alpha = 1$. Then $\sum_{n=1}^{\infty} E \|Y_n\| < \infty$. We have $\sum_{n=1}^{\infty} (X_n)/(n^{1/r})$ converges a.s. By Kronecker lemma, $(S_n - nc)/(n^{1/r})$ converges a.s. for any constant c .

If $r = 1$, by Kronecker lemma, $(S_n/n) - (1/n) \sum_{i=1}^n E(X_i I(\|X_i\| \leq n))$ converges a.s. and $E(X_n I(\|X_n\| \leq n)) \rightarrow EX_1$, we have (2.8) holds.

If $r > 1$, we can show that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{-1/r} E \|X_n I(\|X_n\| > n^{1/r})\| \\
&\leq \sum_{n=1}^{\infty} n^{-1/r} E (\|X_1\| I(\|X_1\| > n^{1/r})) = \sum_{n=1}^{\infty} n^{-1/r} \sum_{j=n+1}^{\infty} \int_{A_j} \|X_1\| dp \\
&= \sum_{j=2}^{\infty} \sum_{n=1}^{j-1} n^{-1/r} \int_{A_j} \|X_1\| dp \leq \frac{r}{r-1} \sum_{j=1}^{\infty} (j-1)^{(r-1)/r} \int_{A_j} \|X_1\| dp \\
&\leq \frac{r}{r-1} \sum_{j=1}^{\infty} \int_{A_j} \|X_1\|^r dp = \frac{r}{r-1} E \|X_1\|^r < \infty.
\end{aligned} \tag{2.14}$$

Therefore, $\sum_{n=1}^{\infty} (X_n - EX_1)/n^{1/r}$ converges a.s. We have (2.8) holds by Kronecker lemma. $\square$

3. The convergence of the weighted sums. Throughout this section, we deal with the almost sure convergence of randomly weighted sums $\sum_{i=1}^{\infty} A_{n,i} X_i$, where $\{X_n : n = 1, 2, 3, \dots\}$ is a sequence of independent and identically distributed random elements in a space of type p and $\{A_{n,i} : n, i = 1, 2, 3, \dots\}$ is an array of random variables satisfying some conditions.

THEOREM 2. Let $\mathbf{B}$ be a separable Banach space of type p . Let $\{X_n : n = 1, 2, 3, \dots\}$ be a sequence of independent and identically distributed random elements in $\mathbf{B}$ such that $E\|X_1\|^r < \infty$ and $r < p$. Moreover, we assume that $EX_1 = 0$ when $r \geq 1$. Let $\{A_{n,i} : n, i = 1, 2, 3, \dots\}$ be an array of random variables such that $\{A_{n,i}\}$ and $\{X_i\}$ are independent and satisfying

$$A_{n,i} = O(i^{-1/r}) \quad \text{a.s. for every } n, \quad (3.1)$$

$$\lim_{n \rightarrow \infty} A_{n,i} = 0 \quad \text{a.s. for every } i, \quad (3.2)$$

$$\sum_{i=1}^{\infty} E|A_{n,i}|^r < \infty \quad \text{for every } n, \quad (3.3)$$

and

$$\sum_{i=1}^{\infty} i^{1/r} |A_{n,i} - A_{n,i+1}| < M \quad \text{a.s. for every } n \text{ and some constant } M > 0. \quad (3.4)$$

Then

$$\lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} A_{n,i} X_i = 0 \quad \text{a.s.} \quad (3.5)$$

Conversely, if

$$\lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} A_{n,i} X_i = 0 \quad \text{a.s.} \quad (3.6)$$

for all arrays $\{A_{n,i}\}$ satisfying the above conditions, then $E\|X_1\|^r < \infty$.

PROOF. Since $\{A_{n,i}\}$ and $\{X_i\}$ are independent, if $r > 1$, we choose $p = q = r$ in Lemma 1, then

$$E \left(\sum_{i=1}^{\infty} \|A_{n,i} X_i\| \right)^r \leq \sum_{i=1}^{\infty} E|A_{n,i}|^r E\|X_1\|^r < \infty. \quad (3.7)$$

If $r \leq 1$, it is obvious that

$$E \left(\sum_{i=1}^{\infty} \|A_{n,i} X_i\| \right)^r \leq \sum_{i=1}^{\infty} E|A_{n,i}|^r E\|X_1\|^r < \infty. \quad (3.8)$$

Therefore, $\sum_{i=1}^{\infty} A_{n,i} X_i$ converges a.s.

Since $A_{n,i} = A_{n,i}^+ - A_{n,i}^-$, without loss of generality, we can assume that $A_{n,i} \geq 0$. Let $S_k = \sum_{i=1}^k X_i$, $Y_k = (S_k/k^{1/r})$ for every $k \geq 1$ and $S_0 = 0$. By Theorem 1, we have $\lim_{k \rightarrow \infty} Y_k = 0$ a.s.

$$\sum_{i=1}^{\infty} A_{n,i} X_i = \sum_{i=1}^{\infty} A_{n,i} (S_i - S_{i-1}) = \lim_{N \rightarrow \infty} \left(\sum_{i=1}^{N-1} (A_{n,i} - A_{n,i+1}) S_i + A_{n,N} S_N \right). \quad (3.9)$$

Since $\{i^{1/r} A_{n,i}\}$ is bounded a.s. for every n and r ,

$$\lim_{N \rightarrow \infty} A_{n,N} S_N = \lim_{N \rightarrow \infty} (N^{1/r} A_{n,N}) Y_N = 0 \quad \text{a.s.} \quad (3.10)$$

We have

$$\sum_{i=1}^{\infty} A_{n,i} X_i = \sum_{i=1}^{\infty} i^{1/r} (A_{n,i} - A_{n,i+1}) Y_i \quad \text{a.s.} \quad (3.11)$$

Let $B_{n,i} = i^{1/r} (A_{n,i} - A_{n,i+1})$. Hence, $\sum_{i=1}^{\infty} |B_{n,i}| \leq M$ a.s. for every n and $\lim_{n \rightarrow \infty} B_{n,i} = 0$ a.s. for every i . Define $D^0 = \{w : \lim_{i \rightarrow \infty} Y_i(w) = 0\}$ and $D_n^1 = \{w : \sum_{i=1}^{\infty} |B_{n,i}(w)| \leq M\}$ for each n and $D_i^2 = \{w : \lim_{n \rightarrow \infty} B_{n,i}(w) = 0\}$ for each i . For every $w \in D^0 \cap \bigcap_{i=1}^{\infty} (D_i^1 \cap D_i^2)$ and every $\epsilon > 0$, we can choose A such that $\|Y_i(w)\| < \epsilon$ for $i \geq A$,

$$\begin{aligned} \sum_{i=1}^{\infty} \|B_{n,i}(w) Y_i(w)\| &\leq \sum_{i=1}^{A-1} |B_{n,i}(w)| \|Y_i(w)\| + \sum_{i=A}^{\infty} |B_{n,i}(w)| \|Y_i(w)\| \\ &\leq \max_{i \leq A-1} \|Y_i(w)\| \sum_{i=1}^{A-1} |B_{n,i}(w)| + M\epsilon \longrightarrow M\epsilon \quad \text{as } n \longrightarrow \infty. \end{aligned} \quad (3.12)$$

Since $\Pr(D^0 \cap \bigcap_{i=1}^{\infty} (D_i^1 \cap D_i^2)) = 1$, $\lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} \|B_{n,i} Y_i\| = 0$ a.s. Therefore,

$$\lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} A_{n,i} X_i = \lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} B_{n,i} Y_i = 0 \quad \text{a.s.} \quad (3.13)$$

If $\lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} A_{n,i} X_i = 0$ a.s. for all arrays $\{A_{n,i}\}$ satisfying the above conditions, we can choose

$$A_{n,i} = \begin{cases} n^{-1/r} & \text{if } i \leq n, \\ 0 & \text{if } i > n. \end{cases} \quad (3.14)$$

Then (2.8) holds. By Theorem 1, we have $E\|X_1\|^r < \infty$. $\square$

REMARK 1. The following example claims that condition (3.4) cannot be omitted. Consider the real number space R as a space of type 2. Choose a sequence $\{X_n : n = 1, 2, 3, \dots\}$ of independent and identically distributed random variables with $EX_1^2 < \infty$ and $EX_1 = 0$. Define

$$A_{n,i} = \begin{cases} n^{-1/2} & \text{if } i \leq n, \\ 0 & \text{if } i > n. \end{cases} \quad (3.15)$$

Choose any $r < 2$ so that condition (3.4) does not hold. By the Central Limit Theorem, $\sum_{i=1}^{\infty} A_{n,i} X_i$ cannot converge to 0 a.s.

Choi and Sung [2] considered the almost sure convergence of $\sum_{i=1}^{\infty} a_{n,i} X_i$ for triangular array of constants. Their Theorem 3 can be regarded as a corollary of Theorem 2.

COROLLARY 1 (Choi and Sung [2, Theorem 3]). *Let $\{X_n : n = 1, 2, 3, \dots\}$ be independent and identically distributed random variables with $EX_1 = 0$ and $E|X_1|^r < \infty$ for some $1 \leq r < 2$. Let $\{a_{n,i} : i = 1, 2, \dots, n; n = 1, 2, 3, \dots\}$ be a triangular array of constants satisfying $\sum_{i=1}^n |a_{n,i} - a_{n,i+1}| = O(n^{-1/r})$, where $a_{n,n+1} = 0$. Then*

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n a_{n,i} X_i = 0 \quad \text{a.s.} \quad (3.16)$$

PROOF. By Theorem 2, we must show that there is a constant $M > 0$ such that

$$\lim_{n \rightarrow \infty} a_{n,i} = 0 \quad \text{for every } i \quad (3.17)$$

and

$$\sum_{i=1}^{\infty} i^{1/r} |a_{n,i} - a_{n,i+1}| < M \quad \text{for every } n. \quad (3.18)$$

There is a constant $C > 0$ such that $\sum_{i=1}^n |a_{n,i} - a_{n,i+1}| \leq Cn^{-1/r}$. We have $|a_{n,i}| \leq Cn^{-1/r}$ for every i . So, $\lim_{n \rightarrow \infty} a_{n,i} = 0$ for every i .

Therefore,

$$\begin{aligned} \sum_{i=1}^{\infty} i^{1/r} |a_{n,i} - a_{n,i+1}| &= \sum_{i=1}^n i^{1/r} |a_{n,i} - a_{n,i+1}| \\ &\leq n^{1/r} \sum_{i=1}^n |a_{n,i} - a_{n,i+1}| \leq C. \end{aligned} \quad (3.19)$$

So, the proof is complete. $\square$

The assumptions of $\{A_{n,i}\}$ in Theorem 2 can be simplified as in Theorem 3 for $r < 1$ and Theorem 4 for $r \geq 1$.

LEMMA 4. *Let $\{b_n : n = 1, 2, 3, \dots\}$ be a sequence of positive numbers. If $\sum_{i=1}^{\infty} i|b_i - b_{i+1}| < \infty$ and $\sum_{i=1}^{\infty} b_i < \infty$, then there exists $C > 0$ such that $ib_i < C$ for all i .*

PROOF. Since $\sum_{i=1}^{\infty} i|b_i - b_{i+1}| < \infty$, there exists $N > 0$ such that $\sum_{k=N}^{\infty} i|b_i - b_{i+1}| < 1$. If the result of this Lemma is false, then any $n, l > 0$, there exists $i > l$ such that $b_i > n/i$. We define

$$n_i \equiv \inf \left\{ i : i > 2n_{j-1} \text{ and } b_i > \frac{j}{i} \right\} \quad \text{if } j \geq 2. \quad (3.20)$$

And

$$n_0 \equiv 0, \quad n_1 \equiv \inf \left\{ i : i > N \text{ and } b_i > \frac{1}{i} \right\}. \quad (3.21)$$

We see that

$$\sum_{i=1}^{\infty} b_i = \sum_{i=1}^{\infty} \sum_{k=n_{i-1}+1}^{n_i} b_k = \sum_{i=1}^{\infty} \sum_{k=n_{i-1}+1}^{n_i} (b_{n_i} + (b_k - b_{n_i})). \quad (3.22)$$

If $m > n \geq N$, then

$$\sum_{i=n}^m b_i - b_m \leq \sum_{i=n}^m \sum_{k=i}^{m-1} |b_k - b_{k+1}| = \sum_{k=n}^{m-1} \sum_{i=n}^k |b_k - b_{k+1}| \leq \sum_{k=n}^{m-1} K |b_k - b_{k+1}| < 1. \quad (3.23)$$

Therefore,

$$\begin{aligned} \sum_{i=1}^{\infty} b_i &= \sum_{i=1}^{\infty} \sum_{k=n_{i-1}+1}^{n_i} (b_{n_i} + (b_k - b_{n_i})) \\ &\geq \sum_{i=2}^{\infty} \left(\frac{i}{n_i} (n_i - n_{i-1}) - 1 \right) \geq \sum_{i=2}^{\infty} \left(\frac{i}{n_i} \times \frac{n_i}{2} - 1 \right) = \infty. \end{aligned} \quad (3.24)$$

But $\sum_{i=1}^{\infty} b_i < \infty$ and the proof is complete. $\square$

When $r < 1$, Theorem 2 can be rewritten as follows.

THEOREM 3. Let $\mathbf{B}$ be a separable Banach space of type p . Let $\{X_n : n = 1, 2, 3, \dots\}$ be a sequence of independent and identically distributed random elements in $\mathbf{B}$ such that $E\|X_1\|^r < \infty$ and $r < 1$. Let $\{A_{n,i} : n, i = 1, 2, 3, \dots\}$ be an array of random variables such that $\{A_{n,i}\}$ and $\{X_i\}$ are independent, and satisfying

$$\lim_{n \rightarrow \infty} A_{n,i} = 0 \quad \text{a.s. for every } i, \quad (3.25)$$

$$\sum_{i=1}^{\infty} E|A_{n,i}|^r < \infty \quad \text{a.s. for every } n, \quad (3.26)$$

and

$$\sum_{i=1}^{\infty} i|A_{n,i} - A_{n,i+1}|^r < M \quad \text{a.s. for every } n \text{ and some constant } M > 0. \quad (3.27)$$

Then $\lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} A_{n,i} X_i = 0$ a.s.

Conversely, if $\lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} A_{n,i} X_i = 0$ a.s. for all arrays $\{A_{n,i}\}$ satisfying the above conditions, then $E\|X_1\|^r < \infty$.

PROOF. Since $A_{n,i} = A_{n,i}^+ - A_{n,i}^-$, without loss of generality, we can assume that $A_{n,i} \geq 0$. We consider $A_{n,i}^r = b_i$ in Lemma 4. Since $\sum_{i=1}^{\infty} i|A_{n,i}^r - A_{n,i+1}^r| \leq \sum_{i=1}^{\infty} i|A_{n,i} - A_{n,i+1}|^r$, for $r < 1$, we have $A_{n,i} = O(i^{-1/r})$ a.s. for every n . From the proof of Theorem 2, we have $\lim_{i \rightarrow \infty} Y_i = 0$ a.s. and $\sum_{i=1}^{\infty} A_{n,i} X_i = \sum_{i=1}^{\infty} i^{1/r} (A_{n,i} - A_{n,i+1}) Y_i$ a.s., where $Y_i = (1/i^{1/r}) \sum_{j=1}^i X_j$.

Define $B_{n,i} = i^{1/r} (A_{n,i} - A_{n,i+1})$. Hence, $\sum_{i=1}^{\infty} |B_{n,i}|^r \leq M$ a.s. and $\lim_{n \rightarrow \infty} B_{n,i} = 0$ a.s. Let $D = D^0 \cap \bigcap_{i=1}^{\infty} (D_i^1 \cap D_i^2)$, where the definitions of D^0 , D_i^1 , and D_i^2 are the same as in Theorem 2. For every $w \in D$ and every $\epsilon > 0$, we can choose A such that $\|Y_i(w)\| < \epsilon$ for $i \geq A$,

$$\begin{aligned} \sum_{i=1}^{\infty} \|B_{n,i} Y_i(w)\|^r &\leq \sum_{i=1}^{A-1} |B_{n,i}|^r \|Y_i(w)\|^r + \sum_{i=A}^{\infty} |B_{n,i}|^r \|Y_i(w)\|^r \\ &\leq \max_{i \leq A-1} \|Y_i(w)\|^r \sum_{i=1}^{A-1} |B_{n,i}|^r + M\epsilon \rightarrow M\epsilon \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (3.28)$$

Since $\Pr(D) = 1$, $\lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} \|B_{n,i} Y_i\|^r = 0$ a.s. Therefore,

$$\lim_{n \rightarrow \infty} A_{n,i} X_i = \lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} B_{n,i} Y_i = 0 \quad \text{a.s.} \quad (3.29)$$

The proof of the converse part is the same as the proof of Theorem 2. So the proof is complete. $\square$

When $r \geq 1$, we can obtain the following theorem:

THEOREM 4. Let $\mathbf{B}$ be a separable Banach space of type p . Let $\{X_n : n = 1, 2, 3, \dots\}$ be a sequence of independent and identically distributed random elements in $\mathbf{B}$ such that $EX_1 = 0$ and $E\|X_1\|^r < \infty$ for $1 \leq r < p$. Let $\{A_{n,i} : n, i = 1, 2, 3, \dots\}$ be an array of

random variables such that $\{A_{n,i}\}$ and $\{X_i\}$ are independent, and satisfying

$$\lim_{n \rightarrow \infty} A_{n,i} = 0 \quad \text{a.s. for every } i, \quad (3.30)$$

$$\sum_{i=1}^{\infty} E|A_{n,i}|^r < \infty \quad \text{for every } n, \quad (3.31)$$

and

$$\sum_{i=1}^{\infty} i^{1/r} |A_{n,i}^r - A_{n,i+1}^r|^{1/r} < M \quad \text{a.s. for every } n \text{ and some constant } M > 0. \quad (3.32)$$

Then

$$\lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} A_{n,i} X_i = 0 \quad \text{a.s.} \quad (3.33)$$

Conversely, if

$$\lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} A_{n,i} X_i = 0 \quad \text{a.s.} \quad (3.34)$$

for all arrays $\{A_{n,i}\}$ satisfying the above conditions, then $E\|X_1\|^r < \infty$.

PROOF. We see that

$$\sum_{i=1}^{\infty} i^{1/r} |A_{n,i} - A_{n,i+1}| < \sum_{i=1}^{\infty} i^{1/r} |A_{n,i}^r - A_{n,i+1}^r|^{1/r} < M \quad (3.35)$$

and

$$\sum_{i=1}^{\infty} i^{1/r} |A_{n,i}^r - A_{n,i+1}^r|^{1/r} < \infty \implies \sum_{i=1}^{\infty} i |A_{n,i}^r - A_{n,i+1}^r| < \infty \quad (\text{since } r \geq 1). \quad (3.36)$$

So, from the proofs of Theorem 2 and Theorem 3, we can obtain this theorem. $\square$

Now, we consider a very special case of $\{A_{n,i}\}$. Let $A_{n,i} = n^{-1/r}$ for $i = 1, 2, \dots, n$ and $A_{n,i} = 0$ for $i > n$. The assumptions of $\{A_{n,i}\}$ in Theorem 4 can be easily verified. Therefore, the Marcinkiewicz-Zygmund law of large numbers in a space of type p can be obtained as the following corollary.

COROLLARY 2. Let $\mathbf{B}$ be a separable Banach space of type p and $\{X_n : n = 1, 2, 3, \dots\}$ be a sequence of independent and identically distributed random elements in $\mathbf{B}$ with zero means. For any $1 \leq r < p$, we have if $E\|X_1\|^r < \infty$, then

$$\left(\frac{1}{n}\right)^{1/r} \sum_{i=1}^n X_i \rightarrow 0 \quad \text{a.s.} \quad (3.37)$$

PROOF. Let $A_{n,i} = n^{-1/r}$ for $i = 1, 2, \dots, n$ and $A_{n,i} = 0$ for $i > n$. Since

$$\lim_{n \rightarrow \infty} A_{n,i} = \lim_{n \rightarrow \infty} n^{-1/r} = 0, \quad (3.38)$$

$$\sum_{i=1}^{\infty} E|A_{n,i}|^r = \sum_{i=1}^n \frac{1}{n} = 1, \quad (3.39)$$

and

$$\sum_{i=1}^{\infty} i^{1/r} |A_{n,i}^r - A_{n,i+1}^r|^{1/r} = n^{1/r} \cdot n^{-1/r} = 1, \quad (3.40)$$

the proof is complete by Theorem 4. $\square$

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