CONVEX AND STARLIKE CRITERIA

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ABSTRACT. We investigate an expression involving the quotient of the analytic representations of convex and starlike functions. Sufficient conditions are found for functions to be starlike of a positive order and convex.

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1. Introduction. Let $S$ denote the class of functions $f$ normalized by $f(0) = f'(0) - 1 = 0$ that are analytic and univalent in the unit disk $\Delta = \{z : |z| < 1\}$. A function $f$ in $S$ is said to be starlike of order $\alpha$, $0 \leq \alpha < 1$, and is denoted by $S^*(\alpha)$ if $\text{Re}\{zf''(z)/f'(z)\} > \alpha$, $z \in \Delta$, and is said to be convex and is denoted by $K$ if $\text{Re}\{1 + zf''(z)/f'(z)\} > 0$, $z \in \Delta$. Mocanu [9] studied linear combinations of the representations of convex and starlike functions and defined the class of $\alpha$-convex functions. In [8], it was shown that if

$$\text{Re}\left[\alpha(1 + zf''(z)/f'(z)) + (1 - \alpha)zf'(z)/f(z)\right] > 0 \quad (1.1)$$

for $z \in \Delta$, then $f$ is starlike for $\alpha$ real and convex for $\alpha \geq 1$.

In this note, we investigate the properties of functions defined in terms of the quotient of the analytic representations of convex and starlike functions. In particular, we consider the class $G_b$ consisting of normalized functions $f$ defined by

$$G_b = \left\{ f : \left| \frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} - 1 \right| < b, \ z \in \Delta \right\}. \quad (1.2)$$

We determine sharp values of $b$ for which $G_b \subset S^*(\alpha), 1/2 \leq \alpha < 1$, and also find values of $b$ for which $G_b \subset K$. It is known ([7, 10]) that $K \subset S^*(1/2)$. We show that $G_1 \subset S^*(1/2) - K$. We also find values of $b$ for which $G_b$ is not starlike and not univalent.

We make use of the following lemma obtained by Jack in [4].

**Lemma A.** Suppose $\omega$ is analytic for $|z| \leq r$, $\omega(0) = 0$ and $|\omega(z)| = \max_{|z|=r} |\omega(z)|$. Then $z_0\omega'(z_0) = k\omega(z_0)$, $k \geq 1$.

2. Main results

**Theorem 1.** If $0 < b \leq 1$ and $G_b$ is defined by (1.2), then $G_b \subset S^*(2/\left(1 + \sqrt{1 + 8b}\right))$. The result is sharp for all $b$.

We prove this theorem in an equivalent form, which we write as
**Theorem 1a.** Set $b = (1 - \alpha)/2\alpha^2, 1/2 \leq \alpha < 1$. Then $G_b \subset S^*(\alpha)$, with extremal function $z/(1 - z)^2(1 - \alpha)$.

**Proof of Theorem 1a.** It is well known that if $\omega(z)$ is analytic in $\Delta$ with $\omega(0) = 0$, then $\Re \left( \frac{1 + (1 - 2\alpha)\omega(z)}{1 - \omega(z)} \right) > \alpha$, $z \in \Delta$, if and only if $\omega(z)$ is a Schwarz function, i.e., $|\omega(z)| < 1$ for $z \in \Delta$ with $\omega(0) = 0$. Set

$$p(z) = \frac{zf'(z)}{f(z)} = \frac{1 + (1 - 2\alpha)\omega(z)}{1 - \omega(z)}$$

Then

$$1 + \frac{zf''(z)}{f'(z)} = p(z) + \frac{zp'(z)}{p(z)}$$

and

$$\left| \frac{1 + zf''(z)}{f'(z)} - 1 \right| = \left| \frac{zp'(z)}{p(z)} \right| = \left| \frac{2(1 - \alpha)z\omega(z)}{(1 + (1 - 2\alpha)\omega(z))^2} \right|.$$ (2.3)

If $f \notin S^*(\alpha)$, then by Lemma A there is a $z_0 \in \Delta$ for which $|\omega(z_0)| = 1$ and $z_0\omega'(z_0) \geq \omega(z_0)$. It then follows from (2.3) that $\left| \frac{z_0p'(z_0)}{(p(z_0))^2} \right| \geq \frac{2(1 - \alpha)}{(2\alpha)^2}$ which contradicts our hypothesis. This completes the proof.

**Corollary 1.** $G_1 \subset S^*(1/2)$.

**Proof.** Set $b = 1$ in Theorem 1.

**Corollary 2.** If $\Re \left( \frac{zf'(z)}{1 + zf''(z)} \right) > 1/2$ for $z \in \Delta$, then $f \in S^*(1/2)$.

**Proof.** This follows from Corollary 1 upon noting that for any complex value $w$, $|w - 1| < 1 \iff \Re (1/w) > 1/2$.

We next give a partial converse to Corollary 1.

**Theorem 2.** If $f \in S^*(1/2)$, then $\left| \left( \frac{1 + zf''(z)}{zf'(z)} \right) - 1 \right| < 1$ for $|z| < (2\sqrt{3} - 3)^{1/2} = 0.68 \ldots$. The result is sharp.

**Proof.** Set $p(z) = zf'(z)/f(z) = 1/(1 - \omega(z))$, where $\omega(z)$ is a Schwarz function. We need to find the largest disk $|z| < R$ for which $|zp'(z)/p(z)|^2 = |z\omega'(z)| < 1$. Dieudonné [2] found the region of values for the derivative of Schwarz functions. This led to the sharp bound [3],

$$|\omega'(z)| \leq \begin{cases} 1, & r = |z| \leq \sqrt{2} - 1 \\ \frac{(1 + r^2)^2}{4r(1 - r^2)}, & r \geq \sqrt{2} - 1. \end{cases}$$ (2.4)

Since $|z\omega'(z)| \leq (1 + r^2)^2/4(1 - r^2) = 1$ for $r = (2\sqrt{3} - 3)^{1/2}$, the proof is complete.

**3. A Counterexample.** The extreme points of the closed convex hull of convex functions and functions starlike of order 1/2 are identical. See [1]. Since $G_1 \subset S^*(1/2)$, one might, also, expect to have $G_1 \subset K$. Surprisingly, this is not the case. We now construct a function $f \in G_1 - K$.
THEOREM 3. $G_1 \notin K$.

**Proof.** $G_1 \subset S^+ (1/2)$. Any of $f \in G_1$ satisfies $zf'(z)/f(z) = 1/(1 - \omega(z))$ for some Schwarz function $\omega(z)$. Setting $\alpha = 1/2$ in (2.3), we see that $f \in G_1 \iff |z\omega'(z)| < 1$ for $z \in \Delta$, which means that $z\omega'(z)$ must, also, be a Schwarz function. Since $1 + zf''(z)/f(z) = (1 + z\omega'(z))/(1 - \omega(z))$, it suffices to construct a Schwarz function $\Omega(z) = z\omega'(z)$ for which

$$\text{Re}\left\{\frac{1 + \Omega(z)}{1 - \omega(z)}\right\} < 0$$

(3.1)

at some point $z \in \Delta$. Let

$$A = \{z \in \Delta : |z - z_0| < 10^{-5}, z_0 = e^{\pi i/4} = e^{i\theta_0}\},$$

and set

$$\phi(z) = (z_0 + \bar{z}_0)[(1 - \bar{z}_0 z)^{1/N} - 1],$$

(3.3)

where $N$ is large enough so that $|\phi(z)/z| < 10^{-4}$ for $z \in \Delta - A$ and $|\text{Im} \phi(z)| < 10^{-8}$ for $z \in A$. Define $\Omega$ by $\Omega(z) = 0.9999(z + \phi(z))$.

We first show that $\Omega(z)$ (and, consequently, $\omega(z)$) is a Schwarz function and then show that inequality (3.1) holds when $z = z_0$.

If

$$z \in \Delta - A,$$

(3.4)

then

$$|\Omega(z)| \leq 0.9999(|z| + |\phi(z)|) \leq 0.9999(1.0001) < 1.$$  

(3.5)

If $z \in A$, set $z = z_0 - \epsilon e^{i\theta}$, $0 < \epsilon < 10^{-5}$, and note that $-2 \cos \theta_0 \leq \text{Re} \phi(z) \leq 0$. If $\text{Re} (z + \phi(z)) \geq 0$, then $|z + \text{Re} \phi(z)| \leq |z| < 1$. If $\text{Re} (z + \phi(z)) < 0$, then

$$|z + \text{Re} \phi(z)| \leq \sqrt{(-2 \cos \theta_0 + \epsilon)^2 + (\sin \theta_0 + \epsilon)^2} < \sqrt{1 + 4\epsilon} < 1 + 2\epsilon < 1.0001.$$  

(3.6)

Thus, if $z \in A$,

$$|\Omega(z)| \leq 0.9999|z + \text{Re} \phi(z)| + |\text{Im} \phi(z)| < 0.9999(1.0001) + 10^{-8} = 1.$$  

(3.7)

Therefore, $\Omega(z)$ is a Schwarz function.

We now show that (3.1) holds at $z = z_0$ for this choice of $\Omega(z)$. Since

$$\left|\frac{\Omega(z)}{z} - 1\right| = |\omega'(z) - 1| < 0.0002 \quad \text{for } z \in \Delta - A,$$

(3.8)

we may write $\omega(z) = z + \eta(z)$, where $|\eta(z)| < 0.0003$ for $z \in A$. Note that

$$\left|(1 - \omega(z_0))\right|^2 \text{Re} \left\{\frac{1 + \Omega(z_0)}{1 - \Omega(z_0)}\right\} = \text{Re} \left\{(1 - \Omega(z_0))(1 + \overline{\omega(z_0)})\right\}$$

$$= \text{Re} \left\{(1 - 0.9999\bar{z}_0)(1 - \bar{z}_0 - \overline{\eta(z_0)})\right\}$$

$$\leq 1 - 1.9999\cos \theta_0 + 0.9999\cos 2\theta_0 + 2|\eta(z_0)|$$

$$< 1 - 1.9999\cos (\pi/4) + 0.0006 < 0.$$  

(3.9)

Hence, the function $f$ for which $1 + zf''(z)/f'(z) = (1 + \Omega(z))/(1 - \omega(z))$ must be in $G_1 - K$. 

□
4. Convexity. Since \( G_1 \not\subset K \), we can ask if \( G_b \subset K \) for some \( b < 1 \). In general, \( S^*(\alpha) \not\subset K \) even for \( \alpha \) arbitrary close to 1 (\( b \) close to 0). To see this, we note that \( f_n(z) = z + a_n z^n \) is in \( S^*(\alpha) \) if and only if \( |a_n| \leq (1-\alpha)/(n-\alpha) \) and \( f_n(z) \in K \) if and only if \( |a_n| \leq 1/n^2 \). Thus, \( f(z) = z + (1-\alpha)/(n-\alpha) z^n \in S^*(\alpha) - K \) for \( n > 2/(1-\alpha) \).

We next show that there are values of \( b \) for which the functions in \( G_b \) must be convex.

**Theorem 4.** \( G_b \subset K \) for \( b \leq \sqrt{2}/2 \).

**Proof.** Since \( f \in G_b \subset G_1 \subset S^*(1/2) \), we may write \( z f'(z)/f(z) = 1/(1-\omega(z)) \), where \( \omega \) is a Schwarz function. For \( f \in G_b \), we take \( \alpha = 1/2 \) in (2.3) to obtain \( |zw'(z)| < \sqrt{2}/2 \) and, consequently, \( |\omega(z)| < \sqrt{2}/2 \), \( z \in \Delta \). We need to show that

\[
\text{Re} \left\{ 1 + z f''(z)/f'(z) \right\} = \text{Re} \left\{ (1 + z \omega'(z))/(1 - \omega(z)) \right\} > 0. \tag{4.1}
\]

Since

\[
\left| \arg \left( \frac{1 + z \omega'(z)}{1 - \omega(z)} \right) \right| \leq \left| \arg (1 + z \omega'(z)) \right| + \left| \arg (1 - \omega(z)) \right| 
\leq \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}, \tag{4.2}
\]

the result follows. \( \square \)

In [6], MacGregor found the radius of convexity for \( S^*(1/2) \) to be \( (2\sqrt{3} - 3)^{1/2} = 0.68 \ldots \). Since \( G_1 \subset S^*(1/2) \), we know that the radius of convexity is at least this large. The following consequence of Theorem 4 is that functions in \( G_1 \) are convex in the disk \( |z| < \sqrt{2}/2 \).

**Corollary.** If \( f \in G_b \), \( \sqrt{2}/2 \leq b \leq 1 \), then \( f \) is convex in the disk \( |z| < \sqrt{2}/2b \).

**Proof.** If \( |z \omega'(z)| < 1 \) for \( z \in \Delta \), then \( |z \omega'(z)| < t \) for \( |z| < t < 1 \). If \( f \in G_b \), then \( |z \omega'(z)| < b \) for \( z \in \Delta \). Hence, \( |z \omega'(z)| < \sqrt{2}/2 \) when \( |z| < \sqrt{2}/2b \). \( \square \)

5. Examples. Theorem 1 gives a sharp order of starlikeness for \( G_b \) when \( 0 < b \leq 1 \), with \( G_1 \subset S^*(1/2) \). Our methods do not extend to \( b > 1 \), but we expect the order of starlikeness to decrease from \( 1/2 \) to 0 as \( b \) increases from 1 to some value \( b_0 \) after which functions in \( G_b \) need not be starlike. We do not have a sharp result for \( b > 1 \), but our next example shows that the univalent functions in \( G_b \) are not necessarily starlike for \( b \geq 11.66 \).

The function \( h(z) = z(1 - i z)^{t-1} \) is spiral-like [11] and, hence, in \( S \) because

\[
\text{Re} \left\{ e^{\pi i/4} \frac{zh'(z)}{h(z)} \right\} = \frac{1}{\sqrt{2}} \left( \frac{1 - |z|^2}{|1 - i z|^2} \right) > 0, \quad z \in \Delta. \tag{5.1}
\]

Since \( z h'(z)/h(z) = (1 + z)/(1 - i z) \), we see that \( h \) is not starlike for \( |z| < a, \sqrt{2}/2 < a < 1 \). Thus, \( f(z) = f(a) = h(az)/a \) is not starlike for \( z \in \Delta \). Setting \( p(z) = z f'(z)/f(z) = (1 + az)/(1 - aiz) \), we have

\[
\left| \frac{zp'(z)}{(p(z))^2} \right| = \left| \frac{(1 + iaz)}{(1 + az)^2} \right| \leq \frac{\sqrt{2}a}{(1 - a)^2} < 11.66 \tag{5.2}
\]
for \( a \) sufficiently close to \( \sqrt{2}/2 \). Hence, \( f \in G_b - S^*(0) \) for \( b = 11.66 \).

Finally, we show that the functions in \( G_b \) need not be univalent. In [5], it is shown for \( h(z) = z(1 - iz)^{i-1} \) that \( g(z) = \int_0^z h(t)/t \, dt = (1 - iz)^i - 1 \) is not in \( S \) because \( g(z_0) = g(-z_0) \) for \( z_0 = i(e^{2\pi} - 1)/(e^{2\pi} + 1), |z_0| = 0.996 \). We, thus, conclude that for \( f(z) = g(cz)/c, c = 0.997, f \in G_b - S \) for \( b \) sufficiently large.

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**References**


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