RESEARCH NOTES

A NEW PROOF OF MONOTONICITY FOR EXTENDED MEAN VALUES

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ABSTRACT. In this article, a new proof of monotonicity for extended mean values is given.

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1. Introduction. Stolarsky [14] first defined the extended mean values $E(r, s; x, y)$ and proved that it is continuous on the domain $\{(r, s; x, y) : r, s \in \mathbb{R}, x, y > 0\}$ as follows

$$E(r, s; x, y) = \left(\frac{r}{s} \cdot \frac{y^s - x^s}{y^r - x^r} \right)^{1/(s-r)}, \quad rs(r-s)(x-y) \neq 0; \quad (1.1)$$

$$E(r, 0; x, y) = \left(\frac{y^r - x^r}{\ln y - \ln x} \cdot \frac{1}{r} \right)^{1/r}, \quad r(x-y) \neq 0; \quad (1.2)$$

$$E(r, r; x, y) = e^{-1/r} \left(\frac{x^{x^r}}{y^{y^r}} \right)^{1/(x^r - y^r)}, \quad r(x-y) \neq 0; \quad (1.3)$$

$$E(0, 0; x, y) = \sqrt{xy}, \quad x \neq y; \quad (1.4)$$

$$E(r, s; x, x) = x, \quad x = y. \quad (1.5)$$

It is convenient to write $E(r, s; x, y) = E(r, s) = E(x, y) = E$.

Several authors including Leach and Sholander [2, 3], Páles [6] and Yao and Cao [15] studied the basic properties, monotonicity and comparability of the mean values E . Feng Qi [9] and in collaboration with Qiu-mig Luo [7] further investigated monotonicity of E from new viewpoints. Recently, Feng Qi [7] generalized the extended mean values and the weighted mean values [1, 4, 5] as a new concept of generalized weighted mean values with two parameters, and studied its monotonicity and other properties.

In this note, a new proof of monotonicity for extended mean values is given.

2. Lemmas. Let

$$\begin{aligned} g &= g(t) - g(t; x, y) = y^t - x^t / t, t \neq 0; \\ g(0; x, y) &= \ln y - \ln x. \end{aligned} \quad (2.1)$$

It is easy to see that g can be expressed in integral form as

$$g(t; x, y) = \int_x^y u^{t-1} du, \quad t \in R, \quad (2.2)$$

and

$$g^{(n)}(t) = \int_x^y (\ln u)^n u^{t-1} du, \quad t \in R. \quad (2.3)$$

Therefore, the extended mean values can be represented in terms of g by

$$E(r, s; x, y) = \left(\frac{g(s; x, y)}{g(r; x, y)} \right)^{1/(s-r)}, \quad (r-s)(x-y) \neq 0; \quad (2.4)$$

$$E(r, r; x, y) = \exp \left(\frac{g'_r(r; x, y)}{g(r; x, y)} \right), \quad x-y \neq 0.$$

Set $F = F(r, s) = F(x, y) = F(r, s; x, y) = \ln E(r, s; x, y)$, then F also can be expressed as

$$F(r, s; x, y) = \frac{1}{s-r} \int_r^s \frac{g'_t(t; x, y)}{g(t; x, y)} dt, \quad r-s \neq 0; \quad (2.5)$$

$$F(r, r; x, y) = \frac{g'_r(r; x, y)}{g(r; x, y)}.$$

LEMMA 2.1. Assume that the derivative $f''(t)$ exists on an interval I . If $f(t)$ is an increasing or convex downward function respectively on I , then the arithmetic mean of $f(t)$,

$$\begin{aligned} \phi(r, s) &= \frac{1}{s-r} \int_r^s f(t) dt, \\ \phi(r, r) &= f(r), \end{aligned} \quad (2.6)$$

is also increasing or convex downward respectively with r and s on I .

PROOF. Direct calculation yields

$$\begin{aligned} \frac{\partial \phi(r, s)}{\partial s} &= \frac{1}{(s-r)^2} \left[(s-r)f(s) - \int_r^s f(t) dt \right], \\ \frac{\partial^2 \phi(r, s)}{\partial s^2} &= \frac{(s-r)^2 f'(s) - 2(s-r)f(s) + 2 \int_r^s f(t) dt}{(s-r)^3} \equiv \frac{\varphi(r, s)}{(s-r)^3}, \\ \frac{\partial \varphi(r, s)}{\partial s} &= (s-r)^2 f''(s). \end{aligned} \quad (2.7)$$

In the case of $f'(t) \geq 0$, $\partial \phi(r, s)/\partial s \geq 0$, thus, $\phi(r, s)$ increases with r and s , since $\phi(r, s) = \phi(s, r)$.

In the case of $f''(t) \geq 0$, $\varphi(r, s)$ increases with s . Since $\varphi(r, r) = 0$, it is easy to see that $\partial^2 \phi(r, s)/\partial s^2 \geq 0$ holds. Therefore, $\phi(r, s)$ is convex downward with respect to either r or s , since $\phi(r, s) = \phi(s, r)$. $\square$

LEMMA 2.2. *Let $f, h : [a, b] \rightarrow R$ be integrable functions, both increasing or both decreasing. Furthermore, let $p : [a, b] \rightarrow R$ be an integrable and nonnegative function. Then*

$$\int_a^b p(u)f(u)du \int_a^b p(u)h(u)du \leq \int_a^b p(u)du \int_a^b p(u)f(u)h(u)du. \quad (2.8)$$

If one of the functions of f or h is nonincreasing and the other nondecreasing, then the inequality in (2.8) is reversed.

The inequality (2.8) is called Tchebycheff's integral inequality; for details, see [1, 4].

LEMMA 2.3. *Let $i, j, k \in N$, we have*

$$g^{(2(i+k)+1)}(t; x, y)g^{(2(j+k)+1)}(t; x, y) \leq g^{(2k)}(t; x, y)g^{(2(i+j+k+1))}(t; x, y). \quad (2.9)$$

If $x, y \geq 1$, then

$$g^{(i+k)}(t; x, y)g^{(j+k)}(t; x, y) \leq g^{(k)}(t; x, y)g^{(i+j+k)}(t; x, y). \quad (2.10)$$

If $0 < x, y \leq 1$, then

$$g^{(2i+k+1)}(t; x, y)g^{(2j+k+1)}(t; x, y) \leq g^{(k)}(t; x, y)g^{(2(i+j+1)+k)}(t; x, y); \quad (2.11)$$

$$g^{(2i+k+1)}(t; x, y)g^{(2j+k)}(t; x, y) \geq g^{(k)}(t; x, y)g^{(2(i+j)+k+1)}(t; x, y); \quad (2.12)$$

$$g^{(2i+k)}(t; x, y)g^{(2j+k)}(t; x, y) \leq g^{(k)}(t; x, y)g^{(2(i+j)+k)}(t; x, y). \quad (2.13)$$

PROOF. By Tchebycheff's integral inequality (2.8) applied to the functions $p(u) = (\ln u)^{2k}u^{t-1}$, $f(u) = (\ln u)^{2i+1}$ and $h(u) = (\ln u)^{2j+1}$ for $i, j, k \in N$, $u \in [x, y]$, $t \in R$, inequality (2.9) follows easily.

By the same arguments, inequalities (2.10), (2.11), (2.12), and (2.13) also follow from Tchebycheff's integral inequality. $\square$

LEMMA 2.4. *The functions $g_t^{(2(k+i)+1)}(t; x, y)/g_t^{(2k)}(t; x, y)$ are increasing with respect to t, x , and y for i and k being nonnegative integers.*

PROOF. By simple computation, we have

$$\left(\frac{g^{(2(k+i)+1)}(t)}{g^{(2k)}(t)} \right)' = \frac{g^{(2(i+k+1))}(t)g^{(2k)}(t) - g^{(2(i+k)+1)}(t)g^{(2k+1)}(t)}{[g^{(2k)}(t)]^2}. \quad (2.14)$$

Combining (2.9) and (2.14), we conclude that the derivative of $g^{(2(k+i)+1)}(t)/g^{(2k)}(t)$ with respect to t is nonnegative, and $g^{(2(k+i)+1)}(t; x, y)/g_t^{(2k)}(t; x, y)$ increases with t .

Differentiating directly, using the integral expression (2.3) of g and rearranging gives

$$\begin{aligned}
& \frac{\partial}{\partial y} \left(\frac{g_t^{(2(k+i)+1)}(t; x, y)}{g_t^{(2k)}(t; x, y)} \right) \\
&= \frac{\partial / \partial y \left[g_t^{(2(k+i)+1)}(t; x, y) \right] g_t^{(2k)}(t; x, y) - g_t^{(2(k+i)+1)}(t; x, y) \partial / \partial y \left[g_t^{(2k)}(t; x, y) \right]}{\left[g_t^{(2k)}(t; x, y) \right]^2} \\
&= \frac{y^{t-1} (\ln y)^{2k}}{\left[g_t^{(2k)}(t; x, y) \right]^2} \left[(\ln y)^{2i+1} \int_x^y (\ln u)^{2k} u^{t-1} du - \int_x^y (\ln u)^{2(i+k)+1} u^{t-1} du \right] \geq 0.
\end{aligned} \tag{2.15}$$

Therefore, the desired monotonicity with respect to both x and y follows, for the involved functions are symmetric in x and y . This completes the proof. $\square$

3. Proof of monotonicity

THEOREM 3.1. *The extended mean values $E(r, s; x, y)$ are increasing with respect to both r and s , or to both x and y .*

PROOF. This is a simple consequence of Lemma 2.1 and Lemma 2.3 in combination with its integral forms (2.4) and (2.5) of $E(r, s; x, y)$. $\square$

REMARK 1. It may be pointed out that the method used in this paper could yield more general results (see [4, 12], and so on).

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REFERENCES

- [1] J.-C. Kuang, *Changyong budengshi. [Applied inequalities]*, 2nd ed., Hunan Jiaoyu Chubanshe, Changsha, China, 1993 (Chinese). MR 95j:26001.
- [2] E. B. Leach and M. C. Sholander, *Extended mean values*, Amer. Math. Monthly **85** (1978), 84–90. MR 58 22428. Zbl 379.26012.
- [3] ———, *Extended mean values II*, J. Math. Anal. Appl. **92** (1983), 207–223. MR 85b:26007. Zbl 517.26007.
- [4] D. S. Mitrinović, *Analytic Inequalities*, Springer-Verlag, Berlin, 1970. MR 43#448. Zbl 199.38101.
- [5] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, 1993. MR 94c:00004. Zbl 771.26009.
- [6] Z. Páles, *Inequalities for differences of powers*, J. Math. Anal. Appl. **131** (1988), 271–281. MR 89f:26023. Zbl 649.26014.
- [7] F. Qi, *Generalized weighted mean values with two parameters*, Proc. R. Soc. Lond., Ser. A, Math. Phys. Eng. Sci. **454** (1998), no. 1978, 2723–2732. Zbl 990.09932.
- [8] ———, *On a two-parameter family of nonhomogeneous mean values*, Tamkang Journal of Mathematics **8** (1998), no. 2.
- [9] F. Qi and Q.-M. Luo, *Refinement and extensions of an inequality*, Mathematics and Informatics Quarterly (1998), to appear.
- [10] ———, *A simple proof of monotonicity for extended mean values*, J. Math. Anal. Appl. **224** (1999), no. 2, 356–359. Zbl 980.55030.
- [11] F. Qi and S.-L. Xu, *A new proof of monotonicity for generalized weighted mean values*, submitted.

- [12] ———, *Refinements and extensions of an inequality. II*, J. Math. Anal. Appl. **211** (1997), 616–620. CMP 97 15.
- [13] ———, *The function $(b^x - a^x)/x$: inequalities and properties*, Proc. Amer. Math. Soc. **126** (1998), no. 11, 3355–3359. CMP 97 15.
- [14] K. B. Stolarsky, *Generalizations of the logarithmic mean*, Math. Mag. **48** (1975), 87–92. MR 50 10186. Zbl 302.26003.
- [15] R. Yang and D. Cao, *Generalizations of the logarithmic mean*, J. Ningbo Univ. (1989), no. 2, 105–108.

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