APPLICATIONS OF RUSCHEWEYH DERIVATIVES AND HADAMARD PRODUCT TO ANALYTIC FUNCTIONS

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ABSTRACT. For given analytic functions $\phi(z) = z + \sum_{m=2}^{\infty} \lambda_m z^m$, $\psi(z) = z + \sum_{m=2}^{\infty} \mu_m z^m$ in $U = \{z \mid |z| < 1\}$ with $\lambda_m \geq 0$, $\mu_m \geq 0$ and $\lambda_m \geq \mu_m$, let $E_n(\phi, \psi; A, B)$ be the class of analytic functions $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$ in U such that $(f * \Psi)(z) \neq 0$ and

$$\frac{D^{n+1}(f * \phi)(z)}{D^n(f * \Psi)(z)} \ll \frac{1 + A_z}{1 + B_z}, \quad -1 \leq A < B \leq 1, \quad z \in U,$$

where $D^n h(z) = z(z^{n-1}h(z))^{(n)}/n!$, $n \in N_0 = \{0, 1, 2, \dots\}$ is the n th Ruscheweyh derivative; $\ll$ and $*$ denote subordination and the Hadamard product, respectively. Let T be the class of analytic functions in U of the form $f(z) = z - \sum_{m=2}^{\infty} a_m z^m$, $a_m \geq 0$, and let $E_n[\phi, \psi; A, B] = E_n(\phi, \psi; A, B) \cap T$. Coefficient estimates, extreme points, distortion theorems and radius of starlikeness and convexity are determined for functions in the class $E_n[\phi, \psi; A, B]$. We also consider the quasi-Hadamard product of functions in $E_n[z/(1-z), z/(1-z); A, B]$ and $E_n[z/(1-z)^2, z/(1-z)^2; A, B]$.

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1. Introduction. Let H denote the class of functions $f(z)$ analytic in the unit disc $U = \{z \mid |z| < 1\}$ and normalized by $f(0) = 0$ and $f'(0) = 1$. The Hadamard product of two functions $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$ and $g(z) = z + \sum_{m=2}^{\infty} b_m z^m$ in H is given by

$$(f * g)(z) = z + \sum_{m=2}^{\infty} a_m b_m z^m. \quad (1.1)$$

Let $D^\alpha f(z) = z/(1-z)^{\alpha+1} * f(z)$, $(\alpha \geq -1)$. Ruscheweyh [9] observed that $D^n f(z) = z(z^{n-1}f(z))^{(n)}/n!$ when $n \in N_0 = \{0, 1, 2, \dots\}$. This symbol $D^n f(z)$, $n \in N_0$, was called the n th Ruscheweyh derivative of $f(z)$ by Al-Amiri [2]. Recently, several subclasses of H have been introduced and studied by using either the Hadamard product or Ruscheweyh derivatives (see [1, 4, 7, 8], etc.). To provide a unified approach to the study of various properties of these classes, we introduce the following most generalized subclass of H by using both the Hadamard product and Ruscheweyh derivatives.

DEFINITION 1.1. Given the functions

$$\phi(z) = z + \sum_{m=2}^{\infty} \lambda_m z^m, \quad \psi(z) = z + \sum_{m=2}^{\infty} \mu_m z^m$$

analytic in U such that $\lambda_m \geq 0$, $\mu_m \geq 0$ and $\lambda_m \geq \mu_m$ for $m = 2, 3, \dots$, we say that $f \in H$ is in the class $E_n(\phi, \psi; A, B)$ if $(f * \psi)(z) \neq 0$ and

$$\frac{D^{n+1}(f * \phi)(z)}{D^n(f * \psi)(z)} \ll \frac{1 + Az}{1 + Bz}, \quad z \in U, \quad (1.2)$$

where $\ll$ denote subordination, $-1 \leq A < B \leq 1$ and $n \in N_0$.

Let G be the class of functions w analytic in U and satisfy the conditions $w(0) = 0$ and $|w(z)| < 1$ for $z \in U$. By the definition of subordination, condition (1.2) is equivalent to

$$\frac{D^{n+1}(f * \phi)(z)}{D^n(f * \psi)(z)} = \frac{1 + Aw(z)}{1 + Bw(z)}, \quad w \in G. \quad (1.3)$$

Let T denote the subclass of H consisting of functions of the form $f(z) = z - \sum_{m=2}^{\infty} a_m z^m$, $a_m \geq 0$, and let $E_n[\phi, \psi; A, B] = E_n(\phi, \psi; A, B) \cap T$. It is easy to check that various subclasses of T can be represented as $E_n[\phi, \psi; A, B]$ for suitable choices of $\phi(z)$, $\psi(z)$, A, B , and n . For example,

$$\begin{aligned} E_n \left[\frac{z}{1-z}, \frac{z}{1-z}; A, B \right] &= S_n[A, B], \\ E_n \left[\frac{z}{(1-z)^2}, \frac{z}{(1-z)^2}; A, B \right] &= K_n[A, B], \\ E_0 \left[\frac{z}{(1-z)^{2(1-\gamma)}}, \frac{z}{(1-z)^{2(1-\gamma)}}; (2\alpha-1)\beta, \beta \right] &= R_\gamma[\alpha, \beta], \\ E_0 \left[\frac{z}{(1-z)^{2(1-\gamma)}}, z; (2\alpha-1)\beta, \beta \right] &= P_\gamma[\alpha, \beta], \quad 0 \leq \alpha < 1, 0 < \beta \leq 1, 0 \leq \gamma < 1, \\ E_n \left[\frac{z}{(1-z)}, z; A, B \right] &= V_n[A, B], \end{aligned} \quad (1.4)$$

etc. The classes $S_n[A, B]$ and $K_n[A, B]$ were introduced and studied by Padmanabhan and Manjini [8] whereas $R_\gamma[\alpha, \beta]$, $P_\gamma[\alpha, \beta]$, and $V_n[A, B]$ were, respectively, studied by Ahuja and Silverman [1], Owa and Ahuja [7], and Kumar [4]. Several other subclasses of T , introduced and studied by Silverman [10], Silverman and Silvia [11], Gupta and Jain [3], and others, can also be obtained from the class $E_n[\phi, \psi; A, B]$ by suitably choosing $\phi(z)$, $\psi(z)$, A, B , and n .

Now, we make a systematic study of the class $E_n[\phi, \psi; A, B]$. It is assumed throughout that $\phi(z)$ and $\psi(z)$ satisfy the conditions stated in Definition 1.1 and that $(f * \psi)(z) \neq 0$ for $z \in U$.

2. Coefficient inequalities. In this section, we find a necessary and sufficient condition for a function to be in $E_n[\phi, \psi; A, B]$ and, consequently, calculate coefficient estimates for functions in $E_n[\phi, \psi; A, B]$.

THEOREM 2.1. Let $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$ be in H . If, for some A, B ($-1 \leq A < B \leq 1$),

$$\sum_{m=2}^{\infty} \frac{(m+n-1)! \sigma_m}{(m-1)!(n+1)!} |a_m| \leq B - A, \quad n \in N_0, \quad (2.1)$$

where $\sigma_m = (B+1)(m+n)\lambda_m - (A+1)(n+1)\mu_m$, then $f \in E_n(\phi, \psi; A, B)$.

PROOF. Suppose that condition (2.1) holds for all admissible values of A, B , and n . In view of (1.3), it is sufficient to show that

$$\left| \frac{D^{n+1}(f * \phi)(z) - D^n(f * \psi)(z)}{BD^{n+1}(f * \phi)(z) - AD^n(f * \psi)(z)} \right| < 1, \quad z \in U. \quad (2.2)$$

For $|z| = r$, $0 \leq r < 1$, we have

$$\begin{aligned} & |D^{n+1}(f * \phi)(z) - D^n(f * \psi)(z)| - |BD^{n+1}(f * \phi)(z) - AD^n(f * \psi)(z)| \\ & \leq \sum_{m=2}^{\infty} \frac{(m+n-1)!}{(m-1)!(n+1)!} [(m+n)\lambda_m - (n+1)\mu_m] |a_m| r^m \\ & \quad - \left\{ (B-A)r - \sum_{m=2}^{\infty} \frac{(m+n-1)!}{(m-1)!(n+1)!} [B(m+n)\lambda_m - A(n+1)\mu_m] |a_m| r^m \right\} \\ & < \left[\sum_{m=2}^{\infty} \frac{(m+n-1)!}{(m-1)!(n+1)!} [(B+1)(m+n)\lambda_m - (A+1)(n+1)\mu_m] \right. \\ & \quad \left. \times |a_m| - (B-A) \right] |z| \leq 0, \end{aligned} \quad (2.3)$$

in view of (2.1). Thus, (2.2) is satisfied and, hence, $f \in E_n(\phi, \psi; A, B)$. $\square$

THEOREM 2.2. Let $f \in T$. Then $f \in E_n[\phi, \psi; A, B]$ if and only if (2.1) is satisfied.

PROOF. In view of Theorem 2.1, it is sufficient to show the “only if” part. Thus, let $f \in E_n[\phi, \psi; A, B]$. Then, from (1.3), we get

$$|w(z)| = \left| \frac{\sum_{m=2}^{\infty} \frac{(m+n-1)!}{(m-1)!(n+1)!} [(m+n)\lambda_m - (n+1)\mu_m] |a_m| z^{m-1}}{(B-A) - \sum_{m=2}^{\infty} \frac{(m+n-1)!}{(m-1)!(n+1)!} [B(m+n)\lambda_m - A(n+1)\mu_m] |a_m| z^{m-1}} \right| < 1 \quad (2.4)$$

and, therefore,

$$\operatorname{Re} \left(\frac{\sum_{m=2}^{\infty} \frac{(m+n-1)!}{(m-1)!(n+1)!} [(m+n)\lambda_m - (n+1)\mu_m] |a_m| z^{m-1}}{(B-A) - \sum_{m=2}^{\infty} \frac{(m+n-1)!}{(m-1)!(n+1)!} [B(m+n)\lambda_m - A(n+1)\mu_m] |a_m| z^{m-1}} \right) < 1 \quad (2.5)$$

for all $z \in U$. We consider real values of z and take $z = r$ with $0 < r < 1$. Then, for $r = 0$, the denominator of (2.5) is positive and so is positive for all r , $0 \leq r < 1$. Then (2.5) gives

$$\sum_{m=2}^{\infty} \frac{(m+n-1)!}{(m-1)!(n+1)!} [(B+1)(m+n)\lambda_m - (A+1)(n+1)\mu_m] |a_m| r^{m-1} < B-A. \quad (2.6)$$

Letting $r \rightarrow 1^-$, we get (2.1). $\square$

COROLLARY 2.1. If $f \in E_n[\phi, \psi; A, B]$, then

$$a_m \leq \frac{(m-1)!(n+1)!(B-A)}{(m+n-1)!\sigma_m} \quad \text{for } m = 2, 3, \dots \text{ and } n \in N_0. \quad (2.7)$$

The equality holds, for each m , for functions of the form

$$f_m(z) = z - \frac{(m-1)!(n+1)!(B-A)}{(m+n-1)!\sigma_m} z^m, \quad z \in U. \quad (2.8)$$

REMARK 2.1. Taking different choices of $\phi(z)$, $\psi(z)$, A , B , and n as stated in Section 1, the above theorems lead to necessary and sufficient conditions and, consequently, coefficient inequalities for a function to be in $S_n[A, B]$, $K_n[A, B]$, $R_Y[\alpha, \beta]$, $P_Y[\alpha, \beta]$, $V_n[A, B]$, etc.

3. Closure theorems

THEOREM 3.1. *The class $E_n[\phi, \psi; A, B]$ is closed under convex linear combinations.*

PROOF. Let $f, g \in E_n[\phi, \psi; A, B]$ and let $f(z) = z - \sum_{m=2}^{\infty} a_m z^m$, $g(z) = z - \sum_{m=2}^{\infty} b_m z^m$, $a_m \geq 0$, $b_m \geq 0$. For η such that $0 \leq \eta \leq 1$, it is sufficient to show that the function h , defined by $h(z) = (1 - \eta)f(z) + \eta g(z)$, $z \in U$, belongs to $E_n[\phi, \psi; A, B]$.

Since $h(z) = z - \sum_{m=2}^{\infty} [(1 - \eta)a_m + \eta b_m] z^m$, applying Theorem 2.2, we get

$$\begin{aligned} & \sum_{m=2}^{\infty} \frac{(m+n-1)! \sigma_m}{(m-1)!(n+1)!} [(1-\eta)a_m + \eta b_m] \\ & \leq (1-\eta) \sum_{m=2}^{\infty} \frac{(m+n-1)! \sigma_m}{(m-1)!(n+1)!} a_m + \eta \sum_{m=2}^{\infty} \frac{(m+n-1)! \sigma_m}{(m-1)!(n+1)!} b_m \\ & \leq (1-\eta)(B-A) + \eta(B-A) = (B-A). \end{aligned} \quad (3.1)$$

This implies that $h \in E_n[\phi, \psi; A, B]$. $\square$

From Theorem 3.1 it follows that the closed convex hull of $E_n[\phi, \psi; A, B]$ is the same as $E_n[\phi, \psi; A, B]$. Now, we determine the extreme points of $E_n[\phi, \psi; A, B]$.

THEOREM 3.2. *Let $f_1(z) = z$, $f_m(z) = z - ((m-1)!(n+1)!(B-A)/(m+n-1)!\sigma_m)z^m$, $m = 2, 3, \dots$, $z \in U$, and $n \in N_0$. Then $f \in E_n[\phi, \psi; A, B]$ if and only if it can be expressed as*

$$f(z) = \sum_{m=1}^{\infty} \rho_m f_m(z), \quad \text{where } \rho_m \geq 0 \text{ and } \sum_{m=1}^{\infty} \rho_m = 1. \quad (3.2)$$

PROOF. Suppose that

$$f(z) = \sum_{m=1}^{\infty} \rho_m f_m(z) = z - \sum_{m=2}^{\infty} \rho_m ((m-1)!(n+1)!(B-A)/(m+n-1)!\sigma_m) z^m. \quad (3.3)$$

Since

$$\sum_{m=2}^{\infty} \frac{(m+n-1)! \sigma_m}{(m-1)!(n+1)!(B-A)} \rho_m \frac{(m-1)!(n+1)!(B-A)}{(m+n-1)!\sigma_m} = \sum_{m=2}^{\infty} \rho_m = 1 - \rho_1 \leq 1, \quad (3.4)$$

it follows, from Theorem 2.2, that $f \in E_n[\phi, \psi; A, B]$.

Conversely, suppose that $f(z) = z - \sum_{m=2}^{\infty} a_m z^m \in E_n[\phi, \psi; A, B]$. Since

$$a_m \leq \frac{(m-1)!(n+1)!(B-A)}{(m+n-1)!\sigma_m}, \quad m = 2, 3, \dots, \quad (3.5)$$

we may set

$$\rho_m = \frac{(m+n-1)! \sigma_m}{(m-1)!(n+1)!(B-A)} a_m, \quad m = 2, 3, \dots; n \in N_0 \text{ and } \rho_1 = 1 - \sum_{m=2}^{\infty} \rho_m. \quad (3.6)$$

From Theorem 2.2, we have $\sum_{m=2}^{\infty} \rho_m \leq 1$ and so $\rho_1 \geq 0$. It follows that $f(z) = \sum_{m=1}^{\infty} \rho_m f_m(z)$. $\square$

COROLLARY 3.1. *The extreme points of $E_n[\phi, \psi; A, B]$ are the functions $f_m(z)$, $m = 1, 2, \dots$.*

4. Distortion theorems. With the aid of Theorem 3.2, we may now find bounds on the modulus of $f(z)$ and $f'(z)$ for $f \in E_n[\phi, \psi; A, B]$.

THEOREM 4.1. *Let $f \in E_n[\phi, \psi; A, B]$ and $\sigma_m = (B+1)(m+n)\lambda_m - (A+1)(n+1)\mu_m$, $m = 2, 3, \dots$. If $n, m, \sigma_m, \sigma_{m+1}$ and $|z|$ satisfy the condition*

$$(m+n)\sigma_{m+1} - m\sigma_m |z| \geq 0, \quad (4.1)$$

then

$$\max \left\{ 0, |z| - \frac{B-A}{\sigma_2} |z|^2 \right\} \leq |f(z)| \leq |z| + \frac{B-A}{\sigma_2} |z|^2. \quad (4.2)$$

The bounds are sharp.

PROOF. By virtue of Theorem 3.2, we note that

$$\begin{aligned} |f(z)| &\geq \max \left\{ 0, |z| - \max_m \frac{(m-1)!(n+1)!(B-A)}{(m+n-1)!\sigma_m} |z|^m \right\}, \\ |f(z)| &\leq |z| + \max_m \frac{(m-1)!(n+1)!(B-A)}{(m+n-1)!\sigma_m} |z|^m \end{aligned} \quad (4.3)$$

for $z \in U$. Thus, it suffices to show that

$$J(A, B, n, m, \sigma_m, |z|) = \frac{(m-1)!(n+1)!(B-A)}{(m+n-1)!\sigma_m} |z|^m \quad (4.4)$$

is a decreasing function of m ($m \geq 2$). It is easily seen that, for $|z| \neq 0$,

$$J(A, B, n, m, \sigma_m, |z|) \geq J(A, B, n, m+1, \sigma_{m+1}, |z|) \quad (4.5)$$

if and only if

$$(m+n)\sigma_{m+1} - m\sigma_m |z| \geq 0 \quad (4.6)$$

which is (4.1). Hence,

$$\max_m J(A, B, n, m, \sigma_m, |z|) \quad (4.7)$$

is attained at $m = 2$ and the proof is complete. $\square$

Finally, since the functions $f_m(z)$, $m \geq 2$, defined in Theorem 3.2, are extreme points of the class $E_n[\phi, \psi; A, B]$, we can see that the bounds of the theorem are attained for the function $f_2(z) = z - ((B-A)/\sigma_2)z^2$.

COROLLARY 4.1 [1]. If $f \in R_\gamma[\alpha, \beta]$, $0 \leq \alpha < 1$, $0 < \beta \leq 1$, and either

$$0 \leq \gamma \leq \frac{(2+3\beta-\alpha\beta)}{(2+4\beta-2\alpha\beta)} \quad \text{or} \quad |z| \leq \frac{(1+2\beta-\alpha\beta)}{(1+3\beta-2\alpha\beta)}, \quad (4.8)$$

then

$$\begin{aligned} \max \left\{ 0, |z| - \frac{\beta(1-\alpha)}{(1-\gamma)[1+\beta(3-2\alpha)]} |z|^2 \right\} \\ \leq |f(z)| \leq |z| + \frac{\beta(1-\alpha)}{(1-\gamma)[1+\beta(3-2\alpha)]} |z|^2. \end{aligned} \quad (4.9)$$

The bounds are sharp.

PROOF. Choosing

$$\phi(z) = \psi(z) = \frac{z}{(1-z)^{2(1-\gamma)}} = z + \sum_{m=2}^{\infty} C(\gamma, m) z^m, \quad (4.10)$$

where

$$C(\gamma, m) = \frac{\left(\prod_{k=2}^m (k-2\gamma) \right)}{(m-1)!}, \quad (4.11)$$

so that $\lambda_m = \mu_m = C(\gamma, m)$ together with $A = (2\alpha-1)\beta$, $B = \beta$ and $n=0$ in Theorem 4.1, the bounds (4.2) reduces to (4.9) provided

$$\begin{aligned} mC(\gamma, m+1)[m+\beta(m+2-2\alpha)] \\ - mC(\gamma, m)[m-1+\beta(m+1-2\alpha)]|z| \geq 0. \end{aligned} \quad (4.12)$$

Since

$$C(\gamma, m+1) = \frac{m+1-2\gamma}{m} C(\gamma, m), \quad (4.13)$$

the above inequality reduces to

$$(m+1-2\gamma)[m+\beta(m+2-2\alpha)] - m[m-1+\beta(m+1-2\alpha)]|z| \geq 0. \quad (4.14)$$

Now, proceeding exactly on the lines of Ahuja and Silverman [1], the result follows. $\square$

COROLLARY 4.2 [7]. If $f \in P_\gamma[\alpha, \beta]$, $0 \leq \alpha < 1$, $0 < \beta \leq 1$, and either $0 \leq \gamma \leq 5/6$ or $|z| \leq 3/4$, then

$$\max \left\{ 0, |z| - \frac{\beta(1-\alpha)}{2(1-\gamma)(1+\beta)} |z|^2 \right\} \leq |f(z)| \leq |z| + \frac{\beta(1-\alpha)}{2(1-\gamma)(1+\beta)} |z|^2. \quad (4.15)$$

The bounds are sharp.

PROOF. Taking

$$\phi(z) = \frac{z}{(1-z)^{2(1-\gamma)}} = z + \sum_{m=2}^{\infty} C(\gamma, m) z^m, \quad \psi(z) = z, \quad (4.16)$$

so that $\lambda_m = C(y, m)$ and $\mu_m = 0$ together with $A = (2\alpha - 1)\beta$, $B = \beta$ and $n = 0$ in Theorem 4.1, the bounds (4.2) reduces to (4.15) provided

$$m(m+1)(1+\beta)C(y, m+1) - m^2(1+\beta)C(y, m)|z| \geq 0. \quad (4.17)$$

Using

$$C(y, m+1) = \frac{m+1-2y}{m}C(y, m), \quad (4.18)$$

the above inequality reduces to

$$(m+1)(m+1-2y) - m^2|z| \geq 0. \quad (4.19)$$

Now, proceeding exactly on the lines of Owa and Ahuja [7], the result follows. $\square$

COROLLARY 4.3 [8]. Let $f \in S_n(A, B)$, $-1 \leq A < B \leq 1$ and

$$c_m = (B+1)(m+1) - (A+1)(n+1), \quad m = 2, 3, \dots \quad (4.20)$$

Then

$$\max \left\{ 0, |z| - \frac{B-A}{c_2} |z|^2 \right\} \leq |f(z)| \leq |z| + \frac{B-A}{c_2} |z|^2. \quad (4.21)$$

The bounds are sharp.

PROOF. Choosing $\phi(z) = \psi(z) = z/(1-z) = z + \sum_{m=2}^{\infty} z^m$ in Theorem 4.1 so that $\lambda_m = \mu_m = 1$ for $m \geq 2$, the bounds (4.2) reduces to (4.21) provided

$$\begin{aligned} & (m+n)[(B+1)(m+n+1) - (A+1)(n+1)] \\ & - m[(B+1)(m+n) - (A+1)(n+1)]|z| \geq 0. \end{aligned} \quad (4.22)$$

On simplification, the above inequality becomes

$$\begin{aligned} & m(1-|z|)[(m-1)(B+1) + (n+1)(B-A)] \\ & + (n+1)[m(B+1) + (B-A)n] \geq 0 \end{aligned} \quad (4.23)$$

which is true for all admissible values of m, n, A, B , and $|z|$. Hence, the result follows. $\square$

REMARK 4.1. The bounds for the functions in the classes $K_n[A, B]$ and $V_n[A, B]$ can be similarly deduced from Theorem 4.1 by choosing $\phi(z)$ and $\psi(z)$ suitably as indicated in Section 1.

THEOREM 4.2. Let $f \in E_n[\phi, \psi; A, B]$ and $\sigma_m = (B+1)(m+1)\lambda_m - (A+1)(n+1)\mu_m$, $m = 2, 3, \dots$. If $n, m, \sigma_m, \sigma_{m+1}$, and $|z|$ satisfy the condition

$$(m+n)\sigma_{m+1} - (m+1)\sigma_m|z| \geq 0, \quad (4.24)$$

then

$$\max \left\{ 0, 1 - \frac{2(B-A)}{\sigma_2} |z| \right\} \leq |f'(z)| \leq 1 + \frac{(B-A)}{\sigma_2} |z|. \quad (4.25)$$

The bounds are sharp for the function $f(z) = z - (2(B-A)/\sigma_2)z^2$.

PROOF. By means of Theorem 3.2, we note that

$$\begin{aligned} |f'(z)| &\geq 1 - \max_m \frac{m!(n+1)!(B-A)}{(m+n-1)!\sigma_m} |z|^{m-1}, \\ |f'(z)| &\leq 1 + \max_m \frac{m!(n+1)!(B-A)}{(m+n-1)!\sigma_m} |z|^{m-1} \end{aligned} \quad (4.26)$$

for $z \in U$. Thus, it suffices to show that

$$J^*(A, B, n, m, \sigma_m, |z|) = \frac{m!(n+1)!(B-A)}{(m+n-1)!\sigma_m} |z|^{m-1} \quad (4.27)$$

is a decreasing function of m ($m \geq 2$). But we can see that, for $|z| \neq 0$,

$$J^*(A, B, n, m, \sigma_m, |z|) \geq J^*(A, B, n, m+1, \sigma_{m+1}, |z|) \quad (4.28)$$

if and only if

$$(m+n)\sigma_{m+1} - (m+1)\sigma_m |z| \geq 0 \quad (4.29)$$

which is (4.24). Hence,

$$\max_m J^*(A, B, n, m, \sigma_m, |z|) \quad (4.30)$$

is attained at $m = 2$ and the result follows. $\square$

REMARK 4.2. For suitable choices of $\phi(z)$, $\psi(z)$, A , B , and n as stated in Section 1, the above theorem leads to the corresponding bounds for f' , where f is in $S_n[A, B]$, $K_n[A, B]$, $P_\gamma[\alpha, \beta]$, $R_\gamma[\alpha, \beta]$, $V_n[A, B]$, etc. The different cases can be deduced from Theorem 4.2 as we did in the case of Theorem 4.1 and, hence, we omit the details.

COROLLARY 4.4. Let $f(z) = z - \sum_{m=2}^{\infty} a_m z^m$ be in the class $E_n[\phi, \psi; A, B]$. Then, $f(z)$ is included in a disc with center at the origin and radius r_1 given by $r_1 = (\sigma_2 + B - A)/\sigma_2$ and $f'(z)$ is included in a disc with center at the origin and radius r_2 given by $r_2 = [\sigma_2 + 2(B - A)]/\sigma_2$.

5. Radius of starlikeness and convexity. Padmanabhan and Manjini [8] have shown that the functions in $E_n[\phi, \psi; A, B]$ are starlike in U if $\phi(z) = \psi(z) = z/(1-z)$ and convex in U if $\phi(z) = \psi(z) = z/(1-z)^2$. Now, we determine the largest disc in which functions in $E_n[\phi, \psi; A, B]$ are starlike and convex of order δ ($0 \leq \delta < 1$) in U for all admissible choices of $\phi(z)$, $\psi(z)$, A , B , and n .

THEOREM 5.1. If $f \in E_n[\phi, \psi; A, B]$, then f is starlike of order δ , $0 \leq \delta < 1$ for $|z| < r_1$, where

$$r_1 = \inf_m \left\{ \frac{(m+n-1)!(1-\delta)\sigma_m}{(m-1)!(n+1)!(m-\delta)(B-A)} \right\}^{1/m-1}, \quad (5.1)$$

$m = 2, 3, \dots$, and $n \in N_0$.

PROOF. Let $f \in E_n[\phi, \psi; A, B]$. It is sufficient to show that $|zf'(z)/f(z) - 1| \leq 1 - \delta$ for $|z| < r_1$, where r_1 is as specified in the statement of the theorem. We have

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{\sum_{m=2}^{\infty} (m-1)a_m |z|^{m-1}}{1 - \sum_{m=2}^{\infty} a_m |z|^{m-1}}. \quad (5.2)$$

Thus, $|zf'(z)/(f(z) - 1)| \leq 1 - \delta$ if $\sum_{m=2}^{\infty} ((m - \delta)/(1 - \delta))a_m \leq 1$. By virtue of Theorem 2.2, we only need to find the values of $|z|$ for which the inequality

$$\frac{m - \delta}{1 - \delta} |z|^{m-1} \leq \frac{(m + n - 1)! \sigma_m}{(m - 1)!(n + 1)!(B - A)} \quad (5.3)$$

is valid for all $m = 2, 3, \dots$, which is true when $|z| < r_1$. $\square$

THEOREM 5.2. If $f \in E_n[\phi, \psi; A, B]$, then f is convex of order δ , $0 \leq \delta < 1$ for $|z| < r_2$, where

$$r_2 = \inf_m \left\{ \frac{(m + n - 1)!(1 - \delta)\sigma_m}{m!(n + 1)!(m - \delta)(B - A)} \right\}^{1/m-1}, \quad m = 2, 3, \dots, \text{ and } n \in N_0. \quad (5.4)$$

PROOF. Since $f(z)$ is convex of order δ if and only if $zf'(z)$ is starlike of order δ , the result follows by replacing m with ma_m in Theorem 5.1. $\square$

6. Quasi-Hadamard product. The quasi-Hadamard product of two or more functions has recently been defined and used by several researchers (see [5, 6] etc.). Accordingly the quasi-Hadamard product of $f(z) = z - \sum_{m=2}^{\infty} a_m z^m$, $a_m \geq 0$, and $g(z) = z - \sum_{m=2}^{\infty} b_m z^m$, $b_m \geq 0$, is given by $(f * g)_1(z) = z - \sum_{m=2}^{\infty} a_m b_m z^m$. Choosing $\phi(z) = \psi(z) = z/(1 - z)$ and $\phi(z) = \psi(z) = z/(1 - z)^2$, respectively, in Theorem 2.2, we get the following necessary and sufficient conditions for the functions in $S_n[A, B]$ and $K_n[A, B]$, obtained in [8].

Let $f \in T$. Then $f \in S_n[A, B]$ if and only if

$$\sum_{m=2}^{\infty} \frac{(m + n - 1)! c_m}{(m - 1)!(n + 1)!} a_m \leq B - A, \quad (6.1)$$

and $f \in K_n[A, B]$ if and only if

$$\sum_{m=2}^{\infty} \frac{(m + n - 1)! m c_m}{(m - 1)!(n + 1)!} a_m \leq B - A, \quad (6.2)$$

where $c_m = (B + 1)(m + 1) - (A + 1)(n + 1)$, $n \in N_0$, and $-1 \leq A < B \leq 1$. In this section, we introduce the following new class and establish a theorem concerning the quasi-Hadamard product for functions in $f \in S_n[A, B]$ and $f \in K_n[A, B]$. The theorem and its applications extend the corresponding results obtained by Kumar [5] when $a_{1,i} = 1$, $b_{1,j} = 1$, $i = 1, 2, \dots, p$, $j = 1, 2, \dots, q$.

DEFINITION 6.1. A function $f(z) = z - \sum_{m=2}^{\infty} a_m z^m$, $a_m \geq 0$, which is analytic in U , belongs to the class $S_n^k[A, B]$ if and only if

$$\sum_{m=2}^{\infty} \frac{(m + n - 1)! m^k c_m}{(m - 1)!(n + 1)!} a_m \leq B - A, \quad (6.3)$$

where $c_m = (B+1)(m+n) - (A+1)(n+1)$, $-1 \leq A < B \leq 1$, $n \in N_0$ and k is any fixed nonnegative real number.

It is evident that $S_n^0[A, B] = S_n[A, B]$ and $S_n^1[A, B] = K_n[A, B]$. Further, $S_n^k[A, B] \subset S_n^h[A, B]$ if $k > h \geq 0$, the containment being proper. Whence, for any positive integer k , we have the following inclusion relation:

$$S_n^k[A, B] \subset S_n^{k-1}[A, B] \subset \cdots \subset S_n^2[A, B] \subset K_n[A, B] \subset S_n[A, B]. \quad (6.4)$$

We also note that, for every nonnegative real number k , the class $S_n^k[A, B]$ is nonempty as the functions of the form

$$f(z) = z - \sum_{m=2}^{\infty} \frac{(m-1)!(n+1)!(B-A)}{(m+n-1)!m^k c_m} \xi_m z^m, \quad (6.5)$$

where $\xi_m \geq 0$, $\sum_{m=2}^{\infty} \xi_m \leq 1$, and $n \in N_0$, satisfy the required inequality.

THEOREM 6.1. *Let the functions $f_i(z) = z - \sum_{m=2}^{\infty} a_{m,i} z^m$, $a_{m,i} \geq 0$, belong to the class $K_n[A, B]$ for every $i = 1, 2, \dots, p$ and let the functions $g_j(z) = z - \sum_{m=2}^{\infty} b_{m,j} z^m$, $b_{m,j} \geq 0$, belong to the class $S_n[A, B]$ for every $j = 1, 2, \dots, q$. Then the quasi-Hadamard product $(f_1 * f_2 * \cdots * f_p * g_1 * g_2 * \cdots * g_q)_1(z)$ belongs to the class $S_n^{2p+q-1}[A, B]$.*

PROOF. Since $f_i \in K_n[A, B]$, we have

$$\sum_{m=2}^{\infty} \frac{(m+n-1)!m c_m}{(m-1)!(n+1)!} a_{m,i} \leq B-A \quad (6.6)$$

or

$$a_{m,i} \leq \frac{(m-1)!(n+1)!(B-A)}{(m+n-1)!m c_m} \quad (6.7)$$

for every $i = 1, 2, \dots, p$. The right-hand expression of the last inequality is not greater than m^{-2} for all A, B ($-1 \leq A < B \leq 1$), and $n \in N_0$. Hence,

$$a_{m,i} \leq m^{-2} \quad \text{for every } i = 1, 2, \dots, p. \quad (6.8)$$

Similarly, for $g_j \in S_n[A, B]$, we have

$$\sum_{m=2}^{\infty} \frac{(m+n-1)!c_m}{(m-1)!(n+1)!} b_{m,j} \leq B-A \quad (6.9)$$

and, hence,

$$b_{m,j} \leq m^{-1} \quad \text{for every } j = 1, 2, \dots, q. \quad (6.10)$$

Using (6.8) for $i = 1, 2, \dots, p$; (6.10) for $j = 1, 2, \dots, q-1$; and (6.9) for $j = q$, we get

$$\begin{aligned} & \sum_{m=2}^{\infty} \left[\frac{(m+n-1)!m^{2p+q-1}c_m}{(m-1)!(n+1)!} \prod_{i=1}^p a_{m,i} \prod_{j=1}^q b_{m,j} \right] \\ & \leq \sum_{m=2}^{\infty} \left[\frac{(m+n-1)!m^{2p+q-1}c_m}{(m-1)!(n+1)!} (m^{-2p} m^{-(q-1)}) b_{m,q} \right] \\ & = \sum_{m=2}^{\infty} \left[\frac{(m+n-1)!c_m}{(m-1)!(n+1)!} b_{m,q} \right] \leq B-A. \end{aligned} \quad (6.11)$$

Hence, $(f_1 * f_2 * \cdots * f_p * g_1 * g_2 * \cdots * g_q)_1(z) \in S_n^{2p+q-1}[A, B]$. $\square$

We note that the required estimate can also be obtained by using (6.8) for $i = 1, 2, \dots, p-1$; (6.10) for $j = 1, 2, \dots, q$; and (6.6) for $i = p$.

Taking into account the quasi-Hadamard product of the functions $f_1(z), f_2(z), \dots, f_p(z)$ only in the proof of Theorem 6.1, and using (6.8) for $i = 1, 2, \dots, p-1$; and (6.6) for $i = p$, we are led to the following corollary:

COROLLARY 6.1. *Let the functions $f_i(z) = z - \sum_{m=2}^{\infty} a_{m,i} z^m, a_{m,i} \geq 0$, belong to the class $K_n[A, B]$ for every $i = 1, 2, \dots, p$. Then the quasi-Hadamard product $(f_1 * f_2 * \dots * f_p)_1(z)$ belongs to the class $S_n^{2p-1}[A, B]$.*

Next, taking the quasi-Hadamard product of functions $g_1(z), g_2(z), \dots, g_q(z)$ only in the proof of Theorem 6.1, and using (6.10) for $j = 1, 2, \dots, q-1$; and (6.9) for $j = q$, we get the following corollary:

COROLLARY 6.2. *Let the functions $g_j(z) = z - \sum_{m=2}^{\infty} b_{m,j} z^m, b_{m,j} \geq 0$, belong to the class $S_n[A, B]$ for every $j = 1, 2, \dots, q$. Then the quasi-Hadamard product $(g_1 * g_2 * \dots * g_q)_1(z)$ belongs to the class $S_n^{q-1}[A, B]$.*

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