ON A FOUR-GENERATOR COXETER GROUP

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ABSTRACT. We study one of the 4-generator Coxeter groups and show that it is SQ-universal (SQU). We also study some other properties of the group.

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1. Introduction. We consider the Coxeter group $P$ given by the presentation

$$P = \langle x_1, x_2, x_3, x_4 \mid x_1^2 = x_2^2 = x_3^2 = x_4^2 = (x_1x_2)^3 = (x_2x_3)^3 = (x_1x_3)^3 = (x_1x_4)^3 = (x_3x_4)^3 = (x_2x_3)^3 = e \rangle. \quad (1.1)$$

The Coxeter graph of this group is clearly just a combinatorial tetrahedron:

We observe that each face is the graph of the Euclidean triangle group $\triangle(3, 3, 3)$ which is an affine Weyl group and this contains a nilpotent subgroup of finite index. The group $P$ is infinite and it will be interesting to see its largeness by answering whether it is SQ-universal or not.

2. SQ-universality. We let $S_3$ be the symmetric group of degree 3. Thus

$$S_3 = \langle y_1, y_2 \mid y_1^2 = y_2^2 = (y_1y_2)^3 = e \rangle. \quad (2.1)$$

We consider the map $\theta : P \to S_3$ defined by

$$\theta(x_1) = y_1, \quad \theta(x_2) = y_2, \quad \theta(x_3) = \theta(x_4) = y_1y_2y_1. \quad (2.2)$$

It is easy to see that $\theta$ is an epimorphism and $P/\ker \theta \cong S_3$. A Schreier transversal for $S_3$ in $P$ is $\{ e, x_1, x_2, x_1x_2, x_2x_1, x_1x_2x_1 \}$. A straightforward application of the
Reidemeister-Schreier process gives the following presentation for \( \ker \vartheta \):

\[
\ker \vartheta = \langle a, b, c, d \mid (ad)^3 = (bc)^3 = (abcd)^3 = e \rangle. \tag{2.3}
\]

Letting \( a = d^{-1} \) and \( b = c^{-1} \), we see that \( \ker \vartheta \) is mapped homomorphically onto the free group of rank 2, \( F_2 \). Hence \( \ker \vartheta \) is SQU. Since the index of \( \ker \vartheta \) in \( P \) is finite (6), we get that \( P \) is also SQU [4].

3. The growth series. Let \( (P, X) \) be a Coxeter system and let \( Y \subseteq X \). We denote the subgroup of \( P \), generated by \( Y \), by \( P_Y \). Then \( (W_Y, Y) \) is also a Coxeter system.

In Bourbaki [2, Section 1 of Chapter 4], Exercise 26 gives the following formula for computing the growth series of \( P \) (word growth in the sense of Milner and Gromov):

\[
\sum_{Y \subseteq X} \frac{(-1)^{|Y|}}{P_Y(t)} = \begin{cases} 
\frac{t^m}{P(t)} & \text{if } P \text{ is finite}, \\
0 & \text{if } P \text{ is infinite}. 
\end{cases} \tag{3.1}
\]

In the formula, \( G(t) \) is the growth series of \( G \), \( m \) is the length of the unique element of \( P \) of maximal length.

We use (3.1) to compute \( P(t) \). We compute \( P(t) \) in steps corresponding to the cardinality of \( Y \):

- \(|Y| = 0\) is the trivial subgroup with growth series \( y_0 = 1 \).
- \(|Y| = 1\) four cyclic subgroups of order 2 with growth series \( y_1 = 1 + t \).
- \(|Y| = 2\) six dihedral subgroups of order 6 with growth series \( y_2 = (1 + t)(1 + t + t^2) \).
- \(|Y| = 3\) three affine subgroups with growth series given by \( 1/\gamma - 3/\gamma_1 + 3/\gamma_2 - 1/\gamma_3 = 0 \), that is, \( y_3 = (1 + t + t^2)/(1 - t)^2 \).
- \(|Y| = 4\) the whole group with growth \( y_4(t) = P(t) \) given by \( 1/\gamma - 4/\gamma_1 + 6/\gamma_2 - 4/\gamma_3 + 1/\gamma_4 = 0 \), that is, \( y_4 = (1 + t)(1 + t + t^2)/(1 - t)(1 - t - 3t^2) \).

The growth coefficients \( c_n \) are given by the linear recurrence \( c_0 = 1, c_1 = 4, c_2 = 12, c_3 = 30, c_n = 2c_{n-1} + 2c_{n-2} - 3c_{n-3}, n \geq 4 \) (see [3]). We observe from the growth series \( y_4 \) that zeros of the denominator are not on the unit circle. This implies that \( P \) has no nilpotent subgroup of finite index—in accordance with the fact that \( P \) is SQU.

It is possible to show that the group \( P \) and the Geisking group \( G = \langle x, y \mid x^2 y^2 = xy \rangle \) are isometric and hence \( y_4 \) is also the growth series of \( G \) (see [3]). In [1], it appears that the two Coxeter groups \( T_n \) and \( S_n \) are also isometric and so have the same growth series.

4. The commutator subgroup. Using the Reidemeister-Schreier process, we get the following presentation for \( P' \):

\[
P' = \langle x, y, z \mid x^3 = y^3 = z^3 = (xy)^3 = (xz)^3 = (yz^{-1})^3 = e \rangle. \tag{4.1}
\]

We use \( P' \) to show that \( P \) is SQU in a different method. Let \( K \) be the normal closure of the elements \( xy^{-1}, xz^{-1}, yz^{-1} \) in \( P' \). The group \( K \) has index 3 in \( P' \). Using the Reidemeister-Schreier process, we get the following presentation for \( K \):

\[
K = \langle u_1, u_2, u_3, v_1, v_2, v_3 \mid u_1^2 = v_1^2 = v_2^2 = v_3^2 = u_1 u_2 u_3 \quad = u_1 u_3 u_2 = v_1 v_2 v_3 = u_1 v_2 u_3 v_1 u_2 u_3 = e \rangle. \tag{4.2}
\]
Letting $u_3 = v_3 = e$, we see that $K$ is mapped homomorphically onto $Z \ast Z_3$. Since $Z \ast Z_3$ is SQU (see [4]), therefore $K$ is SQU. Since $K$ is of finite index in $P'$ and $P'$ is of finite index in $P$, we get that $P$ is SQU.

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**References**


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