BANACH-MACKEY, LOCALLY COMPLETE SPACES, AND $\ell_{p,q}$ -SUMMABILITY

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ABSTRACT. We defined the $\ell_{p,q}$ -summability property and study the relations between the $\ell_{p,q}$ -summability property, the Banach-Mackey spaces and the locally complete spaces.

We prove that, for c_0 -quasibarrelled spaces, Banach-Mackey and locally complete are equivalent. Last section is devoted to the study of CS-closed sets introduced by Jameson and Kakol.

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1. Introduction. Let (E, τ) be a locally convex space. If A is absolutely convex its linear span E_A may be endowed with the seminorm topology given by the Minkowski functional of A , we denote it by (E_A, ρ_A) . If A is bounded then (E_A, ρ_A) is a normed space. If every bounded set B is contained in an absolutely convex, closed, bounded set, called a disk A such that (E_A, ρ_A) is complete (barrelled) then E is said to be locally complete (barrelled).

A locally convex space is a Banach-Mackey space if $\sigma(E, E')$ -bounded sets are $\beta(E, E')$ -bounded sets.

Finally, let us define the $\ell_{p,q}$ -summability property. For $1 \leq p \leq \infty$ let q be such that $(1/p) + (1/q) = 1$. A sequence $(x_n)_n \subset E$ is p -absolutely summable if for every ρ continuous seminorm in (E, τ) the sequence $(\rho(x_n))_n$ is in ℓ_p . A p -absolutely summable sequence is $\ell_{p,q}$ -summable if for every $(\lambda_n)_n \in \ell_q$, the series $\sum_{n=1}^{\infty} \lambda_n x_n$ converges to x for some $x \in E$. A locally convex space E has the $\ell_{p,q}$ -summability property if each p -absolutely summable sequence is $\ell_{p,q}$ -summable.

2. $\ell_{p,q}$ -summability. Let $(E, \tau) = (c_0, \sigma(c_0, \ell_1))$. (E, τ) is a locally complete space. Take $\alpha = (\alpha_n)_n \in \ell_1$ and $(e_n)_n$ the canonical unit vectors in c_0 . Then $\rho_\alpha(e_n) = |\alpha_n|$ so $\sum_{n=1}^{\infty} \rho_\alpha(e_n) = \sum_{n=1}^{\infty} |\alpha_n| < \infty$ which means that $(e_n)_n$ is absolutely summable for every continuous seminorm in $\sigma(c_0, \ell_1)$. Now, since $\sum_{n=1}^{\infty} (e_n) \notin c_0$ we have here an example of a space that has the $\ell_{\infty,1}$ -summability property and does not have the $\ell_{1,\infty}$ -summability property.

Now let us establish some properties of the spaces with the $\ell_{p,q}$ -summability property.

THEOREM 2.1. *Let (E, τ) be a locally convex space. If E satisfies the $\ell_{p,q}$ -summability property for $1 \leq p, q \leq \infty$ with $(1/p) + (1/q) = 1$, then E is locally complete.*

PROOF. Let A be a bounded set and $B = \overline{\text{abconv } A}$; B is a disk. Take $(x_n)_n \subset E_B$ a sequence such that $(\rho_B(x_n))_n \in \ell_p$. Since $i : (E_B, \rho_B) \hookrightarrow (E, \tau)$ is continuous, for every continuous seminorm ρ in E , we have $(\rho(x_n))_n \in \ell_p$. So for every $(a_n)_n \in \ell_q$, we have $\sum_{n=1}^{\infty} a_n x_n \rightarrow x$ with respect to τ since E has the $\ell_{p,q}$ -summability property.

Now the sequence of partial sums $\sum_{n=1}^k a_n x_n$ is ρ_B -bounded since it is a ρ_B -Cauchy sequence as we can see

$$\rho_B \left(\sum_{n=1}^{k+r} a_n x_n - \sum_{n=1}^k a_n x_n \right) = \rho_B \left(\sum_{n=k+1}^{k+r} a_n x_n \right) \leq |(a'_n)_n|_q \cdot |(\rho_B(x'_n))_n|_p, \quad (2.1)$$

which is small for k big enough, $(a'_n)_n = (0, \dots, 0, a_{k+1}, \dots, a_{k+r}, 0, \dots)$ and $(x'_n)_n = (0, \dots, 0, x_{k+1}, \dots, x_{k+r}, 0, \dots)$.

So $\{\sum_{n=1}^K a_n x_n : K \in \mathbb{N}\}$ is a ρ_B bounded set in (E_B, ρ_B) .

By [5, Theorem 3.2.4] we have that $(\sum_{n=1}^K a_n x_n)_K$ converges to x in (E_B, ρ_B) . So (E_B, ρ_B) has also the $\ell_{p,q}$ -summability property.

Now, we will prove the space (E_B, ρ_B) is complete. Let $(x_n)_n \subset E_B$ be an absolutely summable sequence with $x_n \neq 0$ for every $n \in \mathbb{N}$, so $(\rho_B(x_n))_n \in \ell_1$ then

$$(\alpha_n)_n = (\rho_B^{1/p}(x_n))_n \in \ell_p, \quad (\beta_n)_n = (\rho_B^{1/q}(x_n))_n \in \ell_q. \quad (2.2)$$

Let $y_n = x_n / \rho_B(x_n)$ then $(y_n)_n$ is ρ_B -bounded. So $(\alpha_n y_n)_n \subset E_B$, $(\rho_B(\alpha_n y_n))_n \in \ell_p$ and $\sum_{n=1}^{\infty} \alpha_n y_n = \sum_{n=1}^{\infty} \alpha_n \beta_n y_n$ converges in (E_B, ρ_B) since (E_B, ρ_B) has the $\ell_{p,q}$ -summability property so (E_B, ρ_B) is a Banach disk. $\square$

COROLLARY 2.2. *Let (E, τ) be a locally convex space. (E, τ) is locally complete if and only if (E, τ) has the $\ell_{\infty,1}$ -summability property.*

PROOF. Let (E, τ) be a locally complete space and $(x_n)_n \subset (E, \tau)$ be a bounded sequence, so there exists a Banach disk $B \subset E$ such that $\{x_n\}_n \subset B$ and $\{x_n\}_n$ is bounded in (E_B, ρ_B) .

Let $(\alpha_n)_n \in \ell_1$, then $(\alpha_n x_n)_n$ is ρ_B -absolutely summable, that is $\sum_{n=1}^{\infty} \rho_B(\alpha_n x_n) < \infty$. Hence $\sum_{n=1}^{\infty} \alpha_n x_n$ converges in (E_B, ρ_B) so it also converges in (E, τ) since $i : (E_B, \rho_B) \hookrightarrow (E, \tau)$ is continuous. So E has the $\ell_{\infty,1}$ -summability property. $\square$

COROLLARY 2.3. *E is a Banach space if and only if E is normed and has the $\ell_{p,q}$ -summability property.*

PROOF. We can reproduce the last part of the proof of Theorem 2.1 to show that E normed and with the $\ell_{p,q}$ -summability property is a locally complete normed space and so a Banach space.

Now suppose E is a Banach space and denote the norm by $\|\cdot\|$. Let $(x_n)_n \subset E$ be a sequence such that $(\|x_n\|)_n \in \ell_p$ and let $(\beta_n)_n \in \ell_q$ then the sequence $(\beta_n x_n)_n$ is absolutely summable that is

$$\sum_{n=1}^{\infty} \|\beta_n x_n\| \leq \left(\sum_{n=1}^{\infty} \|x_n\|^p \right)^{1/p} \left(\sum_{n=1}^{\infty} \|\beta_n\|^q \right)^{1/q} < \infty \quad (2.3)$$

hence summable, since E is a Banach space so E has the $\ell_{p,q}$ -summability property. $\square$

3. Banach-Mackey space

DEFINITION 3.1. E is a c_0 -barrelled (c_0 -quasibarrelled) space if each null sequence in $(E', \sigma(E', E))$ ($(E', \beta(E', E))$) is E -equicontinuous.

Note that a c_0 -barrelled space is a c_0 -quasibarrelled space.

LEMMA 3.2. *If $(E, \mu(E, E'))$ is a Banach-Mackey space, where $\mu(E, E')$ denotes the Mackey topology, and c_0 -quasibarrelled space then it is a c_0 -barrelled space.*

PROOF. Let $A \subset (E', \sigma(E', E))$ be a bounded set, since E is a Banach-Mackey space, E' is also a Banach-Mackey space (cf. [9, Theorem 5, page 158]), and then A is $\beta(E', E)$ -bounded so it is contained in a bounded Banach disk by [2, Observation 8.2.23], since the space is c_0 -quasibarrelled. Then by the same observation we have that $(E', \sigma(E', E))$ is locally complete. $\square$

COROLLARY 3.3. *$(E, \mu(E, E'))$ is c_0 -quasibarrelled and Banach-Mackey if and only if $(E', \sigma(E', E))$ is locally complete.*

PROOF. Necessity follows from previous lemma and [2, Observation 8.2.23]. The other implication follows from the same observation, the note following Definition 3.1 and the fact that by [7, Corollary 3, Theorem 1] we have that $(E', \sigma(E', E))$ locally complete implies $(E, \mu(E, E'))$ is a Banach-Mackey space. $\square$

Following Saxon and Sánchez [8], a space E is dual locally complete if $(E, \sigma(E', E))$ is locally complete; then we can extend the result shown in [8, Theorem 2.6].

COROLLARY 3.4. *$(E, \mu(E, E'))$ is dual locally complete if and only if it is Banach-Mackey and c_0 -quasibarrelled.*

A locally convex space E is quasibarrelled if each barrel that absorbs bounded sets is a neighborhood of zero in E . It is clear that a barrelled space is quasibarrelled, in certain cases they are equivalent.

Note that using [7, Theorem 1] we can easily prove that: a locally convex space E is quasibarrelled and Banach-Mackey if and only if it is a barrelled space. Next proposition summarizes what we know about Banach-Mackey spaces in the case of quasibarrelled spaces.

PROPOSITION 3.5. *Let (E, τ) be a locally convex quasibarrelled space, then the following properties are equivalent:*

- (a) E' is a Banach-Mackey space.
- (b) E is a Banach-Mackey space.
- (c) E is barrelled.
- (d) E' is semireflexive.
- (e) In E' , $\text{abconv } K$ is compact for each $K \subset E'$ compact.

- (f) For every $x_n \rightarrow 0$ in E' and every $(\alpha_n)_n \in \ell_1$, $\sum_{n=1}^{\infty} \alpha_n x_n \rightarrow x$ for some $x \in E'$.
- (g) E' is locally complete.
- (h) E' is locally barrelled.

PROOF. (a) $\Rightarrow$ (b) using [9, Theorem 5, page 158]. (b) $\Rightarrow$ (c) from the previous note. (c) $\Rightarrow$ (d) by [9, Theorem 4, page 153]. (d) $\Rightarrow$ (e) is obtained using the same theorem and the fact that a convex hull of a compact set is totally bounded together with [9, Exercise 5, page 122]. (e) $\Rightarrow$ (f) by [7, Theorems 2 and 3]. (f) $\Rightarrow$ (g) using [3, Proposition III.1.4] and [2, Theorem 5.1.11]. (g) $\Rightarrow$ (h) is trivial. (h) $\Rightarrow$ (a) using [1, Theorem 1]. $\square$

Note that (f) and (g) are equivalent in general, [3, Proposition III.1.4] and [2, Theorem 5.1.11] prove (f) $\Rightarrow$ (g) and do not assume E is quasibarrelled, and the other implication can be obtained using an argument similar to the one in Corollary 2.2.

4. CS-closed sets. In this section, we give a more precise definition of the convex series and their properties, first studied by Jameson [4] and Kkol [6].

DEFINITION 4.1. Let (E, τ) be a locally convex space.

(a) Let $A \subset E$, $(a_n)_n \subset A$ and $(c_n) \subset [0, 1]$ such that $\sum_{n=1}^{\infty} c_n = 1$ if $\sum_{n=1}^{\infty} c_n a_n$ is convergent we say that it is a convex convergent series of elements of A .

(b) $A \subset E$ is CS-closed if each convex convergent series of elements of A belongs to A .

(c) $A \subset E$ is CS-compact if each convex series of elements of A converges to an element of A .

(d) $A \subset E$ is ultrabounded if each convex series of elements of A is convergent in E .

(e) The CS-closure of A is the intersection of all CS-closed sets that contain A .

OBSERVATION. (i) An ultrabounded set is bounded.

(ii) The intersection of CS-closed sets is a CS-closed set.

For convenience let us introduce another definition.

DEFINITION 4.2. (a) $B \subset E$ is called a CS-barrel if it is absolutely convex, absorbent and CS-closed.

(b) E is a locally CS-barrelled (barrelled) space if for each bounded set $A \subset E$ there exists a disk B such that $A \subset B$ and E_B is a CS-barrelled (barrelled) space, that is that each CS-barrel (barrel) is a neighborhood of zero.

Now several properties of barrels also hold for CS-barrels although the last sets are somehow "smaller" than the first sets.

It is clear that if E is a CS-barrelled space then it is a barrelled space.

Now if (E, τ) is locally barrelled, then for each bounded set $A \subset E$ there exists a closed bounded disk B such that $A \subset B \subset E$ and (E_B, ρ_B) is barrelled, so for each CS-barrel U in E_B , $\overline{U}$ is a barrel so it is a zero neighborhood with respect to ρ_B , since (E_B, ρ_B) is metrizable by [4, Theorem 1], U is also a zero neighborhood with respect to ρ_B . So we have proved the following.

PROPOSITION 4.3. (E, τ) is a locally barrelled space if and only if it is locally CS-barrelled space.

The CS-compact hull of a set A is the set of convex convergent series of its elements.

A is CS-compact if each convex series of elements of A converges to an element of A , so we have that the CS-compact hull of a set is not necessarily a CS-compact set. This is the moment to bring in the ultrabounded sets, since the CS-compact hull of an ultrabounded set is a CS-compact set.

PROPOSITION 4.4. *In a locally convex space (E, τ) , CS-barrels absorb ultrabounded sets.*

PROOF. Let W be a CS-barrel and A an ultrabounded set in E . Let D be the balanced CS-compact hull of A , by [6, Corollaries 2–4] D is a Banach disk so E_D is barrelled, and the identity map $i : E_D \rightarrow E$ is continuous so $\overline{W}^\tau \cap E_D$ is a barrel in (E_D, ρ_D) , furthermore it is a neighborhood of zero in E_D , so $A \subset D \subset \lambda \overline{W}^\tau \cap E_D$ for some $\lambda > 0$. Now for $(x_n)_n \subset W \cap E_D$ and $(a_n)_n \in [0, 1]$, with $\sum_n a_n = 1$ such that $\sum_n a_n x_n \rightarrow x$ in (E_D, ρ_D) , since W is a CS-barrel in (E, τ) , we have $\sum_n a_n x_n \rightarrow x$ in (E, τ) and $x \in W$, then $x \in W \cap E_D$ and it is a CS-barrel in (E_D, ρ_D) . By [4, Theorem 1], $W \cap E_D$ and $\overline{W}^\tau \cap E_D$ have the same interior with respect to ρ_B , so $A \subset D \subset \lambda(W \cap E_D) \subset \lambda W$. $\square$

REMARK 4.5. Since every Banach disk is ultrabounded (cf. [6, Proposition 2.2]) then each CS-barrel absorbs Banach disks.

To close this section let us mention that if E is locally barrelled then each CS-barrel is a bornivorous (see [7, proof of Theorem 2(1)]).

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