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EXACT SOLUTIONS OF STEADY PLANE FLOWS USING (r,ψ)-COORDINATES

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ABSTRACT. We determine exact solutions of steady, plane viscous incompressible magnetohydrodynamic (MHD) aligned and non-MHD fluid flows when the polar representation of the streamline patterns for these flows are of the form $(\theta - f(r))/g(r) =$ constant.

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1. Introduction. The objective of this paper is to determine exact solutions of steady, plane viscous incompressible magnetohydrodynamic (MHD) aligned and non-MHD fluid flows when the polar representation of the streamline patterns for these flows are of the form $(\theta - f(r))/g(r) =$ constant. These exact solutions are obtained employing the approach first introduced by Martin [4] for plane viscous flows. This approach involves using a natural curvilinear coordinate system (ϕ,ψ) in the physical plane (x,y), where $\psi =$ constant are the streamlines and $\phi =$ constant is an arbitrary family of curves. Chandna and Labropulu [1] obtained the exact solutions of steady plane viscous flows by taking the arbitrary family of curves $\phi =$ constant to be $x =$ constant. More recently, Labropulu and Chandna [3] studied steady plane MHD aligned flows using this method by setting the arbitrary family of curves $\phi(x,y) =$ constant to be either $\xi(x,y) =$ constant or $\eta(x,y) =$ constant, where $\xi(x,y) + i\eta(x,y)$ is an analytic function of $z = x + iy$.

In this paper, we pose and answer the following two questions:

(i) Given a family of plane curves $(\theta - f(r))/g(r) =$ constant, can fluid flow along these curves?

(ii) Given a family of streamlines $(\theta - f(r))/g(r) =$ constant, what is the exact integral of the flow defined by the given streamline pattern?

To investigate our first question, we assume that fluid flows along the given family of curves $(\theta - f(r))/g(r) =$ constant. Since the streamfunction $\psi(r,\theta) =$ constant as well along these curves, it follows that there exists some function $\gamma(\psi)$ such that

$$\frac{\theta - f(r)}{g(r)} = \gamma(\psi), \quad \gamma'(\psi) \neq 0, \tag{1.1}$$

where $\gamma'(\psi)$ is the derivative of $\gamma(\psi)$.

For this work, the curves $\phi =$ constant are taken to be $r =$ constant. Thus, the (r,ψ)-coordinate system is used. Taking $v_1(r,\theta)$ and $v_2(r,\theta)$ to be the components

of velocity vector field in polar coordinates, we have

$$v_1 = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{1}{rg(r)\gamma'(\psi)}, \qquad v_2 = -\frac{\partial\psi}{\partial r} = \frac{1}{\gamma'(\psi)}\left[\frac{\theta g'(r)}{g^2(r)} + \left(\frac{f(r)}{g(r)}\right)'\right]. \tag{1.2}$$

The plan of this paper is as follows: in Section 2, the equations governing the steady plane motion of infinitely conducting aligned MHD, finitely conducting aligned MHD and non-MHD fluid flows are presented. Section 3 shows the transformation of these equations in the (ϕ,ψ)-net. In Section 4, we outline the method of determining whether a given family of curves can be the streamlines. Section 5 consists of applications of this method.

2. Flow equations. The governing equations of a viscous incompressible and electrically conducting fluid flow, in the presence of a magnetic field, are (see [5])

$$\begin{gathered} \operatorname{div}\vec{v} = 0, \\ \rho(\vec{v}\cdot\operatorname{grad})\vec{v} + \operatorname{grad} p = \mu\nabla^2\vec{v} + \mu^*(\operatorname{curl}\vec{H})\times\vec{H}, \\ \frac{1}{\mu^*\sigma}\operatorname{curl}(\operatorname{curl}\vec{H}) = \operatorname{curl}(\vec{v}\times\vec{H}), \end{gathered} \tag{2.1}$$

where $\vec{v}$ is the velocity vector field, $\vec{H}$ the magnetic field, p the pressure function, and the constants ρ, μ, μ^*, and σ are the fluid density, coefficient of viscosity, magnetic permeability and the electrical conductivity, respectively. The magnetic field $\vec{H}$ satisfies an additional equation

$$\operatorname{div}\vec{H} = 0, \tag{2.2}$$

expressing the absence of magnetic poles in the flow. In the case of aligned (or parallel) flows the magnetic field is everywhere parallel to the velocity field, so that

$$\vec{H} = \beta\vec{v}, \tag{2.3}$$

where β is some unknown scalar function such that

$$\vec{v}\cdot\operatorname{grad}\beta = 0. \tag{2.4}$$

In this paper, we study plane motion in the (x,y)-plane and we define the vorticity function ω, current density function Ω, and energy function h given by

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \qquad \Omega = \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y}, \qquad h = \frac{1}{2}\rho(u^2+v^2) + p, \tag{2.5}$$

where $u(x,y), v(x,y)$ are the velocity components and $H_1(x,y), H_2(x,y)$ the magnetic components. In this work, we deal with both infinitely and finitely conducting fluids. Third equation of system (2.1) is identically satisfied for an infinitely conducting MHD aligned flow. However, this equation requires the current density Ω to be a constant, say Ω_0, for a finitely conducting MHD aligned flow.

2.1. Infinitely conducting flow. Using (2.3) and (2.5) in system (2.1), we find that an infinitely conducting steady plane MHD aligned flow is governed by the following system of six partial differential equations:

$$\begin{aligned}
&\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \quad \text{(continuity)},\\
&\left.\begin{aligned}
&\frac{\partial h}{\partial x}+\mu\frac{\partial \omega}{\partial y}-\rho v\omega+\mu^*\beta v\Omega=0\\
&\frac{\partial h}{\partial y}-\mu\frac{\partial \omega}{\partial x}+\rho u\omega-\mu^*\beta u\Omega=0
\end{aligned}\right\} \quad \text{(linear momentum)},\\
&u\frac{\partial \beta}{\partial x}+v\frac{\partial \beta}{\partial y}=0 \quad \text{(solenoidal)},\\
&\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=\omega \quad \text{(vorticity)},\\
&\beta\omega+v\frac{\partial \beta}{\partial x}-u\frac{\partial \beta}{\partial y}=\Omega \quad \text{(current density)}
\end{aligned} \tag{2.6}$$

for the six functions $u(x,y), v(x,y), h(x,y), \omega(x,y), \Omega(x,y)$, and $\beta(x,y)$. Once a solution of this system is determined, the pressure function $p(x,y)$ and the magnetic vector field $\vec{H}$ are found by using (2.3) and (2.5).

2.2. Finitely conducting and non-MHD flows. A finitely conducting steady plane MHD aligned flow is governed by system (2.6) of six partial differential equations when current density function Ω is replaced by constant Ω_0 in this system.

Ordinary incompressible viscous flow in the absence of external forces is governed by the continuity, the linear momentum and the vorticity equations of system (2.6) when $\Omega = 0$ and $\beta = 0$ are substituted in this system.

3. Alternate formulation. The equation of continuity in system (2.6) implies the existence of a streamfunction $\psi = \psi(x,y)$ such that

$$\frac{\partial \psi}{\partial x}=-v, \qquad \frac{\partial \psi}{\partial y}=u. \tag{3.1}$$

We take $\phi(x,y) =$ constant to be some arbitrary family of curves which generates with the streamlines $\psi(x,y) =$ constant a curvilinear net, so that in the physical plane the independent variables x, y can be replaced by ϕ, ψ.

Let

$$x = x(\phi,\psi), \qquad y = y(\phi,\psi), \tag{3.2}$$

define a curvilinear net in the (x,y)-plane with the squared element of arc length along any curve given by

$$ds^2 = E(\phi,\psi)\,d\phi^2 + 2F(\phi,\psi)\,d\phi\,d\psi + G(\phi,\psi)\,d\psi^2, \tag{3.3}$$

where

$$E=\left(\frac{\partial x}{\partial \phi}\right)^2+\left(\frac{\partial y}{\partial \phi}\right)^2, \qquad F=\frac{\partial x}{\partial \phi}\frac{\partial x}{\partial \psi}+\frac{\partial y}{\partial \phi}\frac{\partial y}{\partial \psi}, \qquad G=\left(\frac{\partial x}{\partial \psi}\right)^2+\left(\frac{\partial y}{\partial \psi}\right)^2. \tag{3.4}$$

Equations (3.2) can be solved to obtain $\phi = \phi(x,y)$, $\psi = \psi(x,y)$ such that

$$\frac{\partial x}{\partial \phi} = J\frac{\partial \psi}{\partial y}, \qquad \frac{\partial x}{\partial \psi} = -J\frac{\partial \phi}{\partial y}, \qquad \frac{\partial y}{\partial \phi} = -J\frac{\partial \psi}{\partial x}, \qquad \frac{\partial y}{\partial \psi} = J\frac{\partial \phi}{\partial x}, \tag{3.5}$$

provided $0 < |J| < \infty$, where J is the transformation Jacobian and,

$$J = \frac{\partial x}{\partial \phi}\frac{\partial y}{\partial \psi} - \frac{\partial x}{\partial \psi}\frac{\partial y}{\partial \phi} = \pm\sqrt{EG - F^2} = \pm W. \tag{3.6}$$

3.1. Infinitely conducting flows. Following Martin [4] and Chandna and Labropulu [1], we transform system (2.6) into the $\phi\psi$-coordinates and we have the following theorem.

Theorem 3.1. *If the streamlines $\psi(x,y)$ = constant of a viscous, incompressible infinitely conducting MHD aligned flow are chosen as one set of coordinate curves in a curvilinear coordinate system ϕ,ψ in the physical plane, then system (2.6) in (x,y)-coordinates may be replaced by the system:*

$$\begin{aligned}
&J\frac{\partial h}{\partial \phi} = \mu\left[F\frac{\partial \omega}{\partial \phi} - E\frac{\partial \omega}{\partial \psi}\right] \\
&J\frac{\partial h}{\partial \psi} = \mu\left[G\frac{\partial \omega}{\partial \phi} - F\frac{\partial \omega}{\partial \psi}\right] + J[\mu^*\beta\Omega - \rho\omega] \qquad \textit{(linear momentum)}, \\
&\frac{\partial}{\partial \psi}\left(\frac{W}{E}\Gamma_{11}^2\right) - \frac{\partial}{\partial \phi}\left(\frac{W}{E}\Gamma_{12}^2\right) = 0 \quad \textit{(Gauss)}, \\
&\beta\omega - \frac{E}{W^2}\frac{\partial \beta}{\partial \psi} = \Omega \quad \textit{(current density)}, \\
&\omega = \frac{1}{W}\left[\frac{\partial}{\partial \phi}\left(\frac{F}{W}\right) - \frac{\partial}{\partial \psi}\left(\frac{E}{W}\right)\right] \quad \textit{(vorticity)}, \\
&\frac{\partial \beta}{\partial \phi} = 0 \quad \textit{(solenoidal)}
\end{aligned} \tag{3.7}$$

of six equations for seven functions E, F, G, h, j, ω, and β of ϕ, ψ.

The Christofell symbols Γ_{11}^2 and Γ_{12}^2 in system (3.7) are given by

$$\Gamma_{11}^2 = \frac{1}{2W^2}\left[-F\frac{\partial E}{\partial \phi} + 2E\frac{\partial F}{\partial \phi} - E\frac{\partial E}{\partial \psi}\right], \qquad \Gamma_{12}^2 = \frac{1}{2W^2}\left[E\frac{\partial G}{\partial \phi} - F\frac{\partial E}{\partial \psi}\right]. \tag{3.8}$$

Using the integrability condition $\partial^2 h/\partial\phi\partial\psi = \partial^2 h/\partial\psi\partial\phi$ in the linear momentum equations of Theorem 3.1, we find that the unknown functions $E(\phi,\psi)$, $F(\phi,\psi)$, $G(\phi,\psi)$, $\Omega(\phi,\psi)$, $\omega(\phi,\psi)$, and $\beta(\psi)$ satisfy the following equations:

$$\omega = \frac{1}{W}\left[\frac{\partial}{\partial \phi}\left(\frac{F}{W}\right) - \frac{\partial}{\partial \psi}\left(\frac{E}{W}\right)\right], \qquad \Omega = \beta\omega - \frac{E}{W^2}\frac{\partial \beta}{\partial \psi}, \tag{3.9}$$

$$\frac{\partial}{\partial \psi}\left(\frac{W}{E}\Gamma_{11}^2\right) - \frac{\partial}{\partial \phi}\left(\frac{W}{E}\Gamma_{12}^2\right) = 0, \tag{3.10}$$

$$\mu W\Delta_2\omega + \frac{\partial}{\partial \phi}[\mu^*\beta\Omega - \rho\omega] = 0, \qquad \frac{\partial \beta}{\partial \phi} = 0, \tag{3.11}$$

where

$$\Delta_2\omega = \frac{1}{W}\left[\frac{\partial}{\partial\phi}\left(\frac{G}{W}\frac{\partial\omega}{\partial\phi} - \frac{F}{W}\frac{\partial\omega}{\partial\psi}\right) + \frac{\partial}{\partial\psi}\left(\frac{E}{W}\frac{\partial\omega}{\partial\psi} - \frac{F}{W}\frac{\partial\omega}{\partial\phi}\right)\right] \tag{3.12}$$

defines the Beltrami's differential operator of second order.

Equations (3.9), (3.10), and (3.11) form an underdetermined system since the coordinate lines ϕ = constant have been left arbitrary. This underdetermined system can be made determinate in a number of different ways and one such possible way is to let $\phi(x,y) = \theta(x,y)$, where (r,θ)-net is the polar coordinate system.

3.2. Finitely conducting and non-MHD flows. Finitely conducting MHD aligned flow in the (ϕ,ψ)-net is governed by (3.9), (3.10), and (3.11) with $\Omega = \Omega_0$, where Ω_0 is a constant.

Ordinary viscous flow is governed by (3.9), (3.10), and (3.11) with $\Omega = 0$, $\beta = 0$, and this reduced system is well studied by Martin [4] and Govindaraju [2].

4. Method. To analyze whether a given family of curves $(\theta - f(r))/g(r)$ = constant can or cannot be the streamlines, we assume the affirmative so that there exists some function $\gamma(\psi)$ such that

$$\frac{\theta - f(r)}{g(r)} = \gamma(\psi), \quad \gamma'(\psi) \neq 0, \tag{4.1}$$

where $\gamma'(\psi)$ is the derivative of the unknown function $\gamma(\psi)$.

Using (4.1), $\phi = r$ and $x = r\cos\theta$, $y = r\sin\theta$, in (3.4), we find that E, F, G, and J are given by

$$\begin{aligned} E &= 1 + r^2[f'(r) + g'(r)\gamma(\psi)]^2, \qquad G = r^2 g^2(r)\gamma'^2(\psi), \\ F &= r^2[f'(r) + g'(r)\gamma(\psi)]g(r)\gamma'(\psi), \qquad J = W = rg(r)\gamma'(\psi). \end{aligned} \tag{4.2}$$

4.1. Infinitely conducting flow. Employing (4.2) and $\phi = r$ in (3.9), (3.10), and (3.11), we find that Gauss' equation (3.10) is identically satisfied and our flow is governed by

$$\begin{aligned} & rg(r)\gamma'(\psi)\frac{\partial^2\omega}{\partial r^2} - 2r[f'(r) + g'(r)\gamma(\psi)]\frac{\partial^2\omega}{\partial r\partial\psi} \\ &\quad + \left[\frac{1 + r^2 f'^2(r)}{rg(r)} + \frac{2rf'(r)g'(r)}{g(r)}\gamma(\psi) + \frac{rg'^2(r)}{g(r)}\gamma^2(\psi)\right]\frac{1}{\gamma'(\psi)}\frac{\partial^2\omega}{\partial\psi^2} \\ &\quad + \left[-f'(r) - rf''(r) + 2r\frac{f'(r)g'(r)}{g(r)} + \left(2r\frac{g'^2(r)}{g(r)} - g'(r) - rg''(r)\right)\gamma(\psi)\right. \\ &\qquad - \left(\frac{1 + r^2 f'^2(\psi)}{rg(r)}\right)\frac{\gamma''(\psi)}{\gamma'^2(\psi)} - \frac{2rf'(r)g'(r)}{g(r)}\frac{\gamma(\psi)\gamma''(\psi)}{\gamma'^2(\psi)} \\ &\qquad \left. - \frac{rg'^2(r)}{g(r)}\frac{\gamma^2(\psi)\gamma''(\psi)}{\gamma'^2(\psi)}\right]\frac{\partial\omega}{\partial\psi} + \left[g(r)\gamma'(\psi) - \frac{\rho}{\mu}\right]\frac{\partial\omega}{\partial r} + \frac{\mu^*}{\mu}\beta(\psi)\frac{\partial\Omega}{\partial r} = 0, \end{aligned} \tag{4.3}$$

$$\begin{aligned}\omega = {} & \left[\frac{1}{r}\frac{f'(r)}{g(r)} + \frac{f''(r)}{g(r)} - \frac{2f'(r)g'(r)}{g^2(r)}\right]\frac{1}{\gamma'(\psi)} \\ & + \left[\frac{1}{r}\frac{g'(r)}{g(r)} + \frac{g''(r)}{g(r)} - \frac{2g'^2(r)}{g^2(r)}\right]\frac{\gamma(\psi)}{\gamma'(\psi)} + \left[\frac{1}{r^2g^2(r)} + \frac{f'^2(r)}{g^2(r)}\right]\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \\ & + \frac{2f'(r)g'(r)}{g^2(r)}\frac{\gamma(\psi)\gamma''(\psi)}{\gamma'^3(\psi)} + \frac{g'^2(r)}{g^2(r)}\frac{\gamma^2(\psi)\gamma''(\psi)}{\gamma'^3(\psi)},\end{aligned} \tag{4.4}$$

$$\Omega = \beta(\psi)\omega - \frac{1+r^2[f'(r)+g'(r)\gamma(\psi)]^2}{r^2g^2(r)\gamma'^2(\psi)}\beta'(\psi) \tag{4.5}$$

of three equations in four unknown functions $\omega, \Omega, \gamma(\psi)$, and $\beta(\psi)$. Equation (4.3), is one equation in two unknown functions when ω and Ω are eliminated using (4.4) and (4.5). Summing up, we have the following theorem.

Theorem 4.1. *If a steady, plane, viscous, incompressible, electrically conducting fluid of infinite electrical conductivity flows along $(\theta - f(r))/g(r)$ = constant in the presence of an MHD aligned field, then the known functions $f(r), g(r)$ and the unknown functions $\beta(\psi), \gamma(\psi)$ must satisfy (4.3), where ω and Ω are given by (4.4) and (4.5), respectively.*

4.2. Finitely conducting and non-MHD flows. In the case of finitely conducting flows, $\Omega = \Omega_0$ is an arbitrary constant. Using (4.4) in (4.3) and (4.5), we get two coupled equations in unknown functions $\gamma(\psi)$ and $\beta(\psi)$ and, therefore, we have the following theorem.

Theorem 4.2. *If a steady, plane, viscous, incompressible, electrically conducting fluid of finite electrical conductivity flows along $(\theta - f(r))/g(r)$ = constant, in the presence of an aligned magnetic field, then the known functions $f(r), g(r)$ and the unknown functions $\beta(\psi), \gamma(\psi)$ must satisfy*

$$\begin{aligned} & rg(r)\gamma'(\psi)\frac{\partial^2\omega}{\partial r^2} - 2r[f'(r)+g'(r)\gamma(\psi)]\frac{\partial^2\omega}{\partial r\partial\psi} \\ & \quad + \left[\frac{1}{rg(r)} + \frac{rf'^2(r)}{g(r)} + \frac{2rf'(r)g'(r)}{g(r)}\gamma(\psi) + \frac{rg'^2(r)}{g(r)}\gamma^2(\psi)\right]\frac{1}{\gamma'(\psi)}\frac{\partial^2\omega}{\partial\psi^2} \\ & \quad + \left[-f'(r) - rf''(r) + 2r\frac{f'(r)g'(r)}{g(r)} + \left(2r\frac{g'^2(r)}{g(r)} - g'(r) - rg''(r)\right)\gamma(\psi)\right. \\ & \qquad - \left(\frac{1}{rg(r)} + \frac{rf'^2(\psi)}{g(r)}\right)\frac{\gamma''(\psi)}{\gamma'^2(\psi)} - \frac{2rf'(r)g'(r)}{g(r)}\frac{\gamma(\psi)\gamma''(\psi)}{\gamma'^2(\psi)} \\ & \qquad \left. - \frac{rg'^2(r)}{g(r)}\frac{\gamma^2(\psi)\gamma''(\psi)}{\gamma'^2(\psi)}\right]\frac{\partial\omega}{\partial\psi} + \left[g(r)\gamma'(\psi) - \frac{\rho}{\mu}\right]\frac{\partial\omega}{\partial r} = 0,\end{aligned} \tag{4.6}$$

$$\beta(\psi)\omega - \frac{1+r^2[f'(r)+g'(r)\gamma(\psi)]^2}{r^2g^2(r)\gamma'^2(\psi)}\beta'(\psi) = \Omega_0, \tag{4.7}$$

where ω is given by (4.4).

In the case of non-MHD fluid flow, $\Omega = \beta = 0$, the known functions $f(r), g(r)$ and the unknown function $\gamma(\psi)$ satisfy (4.6) with ω given by (4.4).

5. Applications. This section deals with various flows to illustrate the method.

Example 5.1 (Flow with $\theta - m_1 r^3 - m_2 r^2 =$ constant as streamlines)**.** We assume that

$$\theta = m_1 r^3 + m_2 r^2 + \gamma(\psi); \quad \gamma'(\psi) \neq 0,\ m_1 \neq 0, \tag{5.1}$$

where $\gamma(\psi)$ is an unknown function of ψ.

Comparing (5.1) with (4.1), we have

$$f(r) = m_1 r^3 + m_2 r^2, \qquad g(r) = 1. \tag{5.2}$$

The streamline pattern for this flow is shown in Figure 5.1.

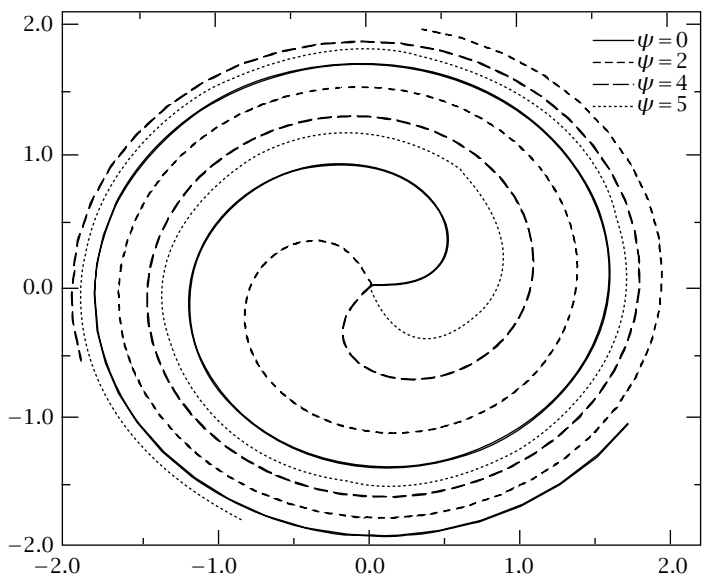

Figure 5.1. Streamline pattern for $\theta - m_1 r^3 - m_2 r^2 = \psi$.

Infinity conducting flow. Employing (4.4) and (4.5) in (4.3), we get

$$\sum_{\substack{n=0,\\ n\neq 1,2}}^{12} A_n(\psi) r^n = 0, \tag{5.3}$$

where

$$A_0(\psi) = \frac{2}{\mu}M_2(\psi) + M_1(\psi) + \frac{4\gamma''(\psi)}{\gamma'^2(\psi)},$$

$$A_3(\psi) = 9m_1 - 6m_1\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)' - \frac{9m_1}{\mu}[\rho - \mu^*\beta^2(\psi)]\frac{1}{\gamma'(\psi)},$$

$$A_4(\psi) = 8m_2^2M_1(\psi) + 32m_2^2\frac{\gamma''(\psi)}{\gamma'^2(\psi)} - \frac{8m_2^2}{\mu}M_2(\psi),$$

$$A_5(\psi) = 24m_1m_2M_1(\psi) + 216m_1m_2\frac{\gamma''(\psi)}{\gamma'^2(\psi)} - \frac{36m_1m_2}{\mu}M_2(\psi),$$

$$A_6(\psi) = 18m_1^2M_1(\psi) - 72m_2^3\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)' + 279m_1^2\frac{\gamma''(\psi)}{\gamma'^2(\psi)} - \frac{36m_1^2}{\mu}M_2(\psi),$$

$$A_7(\psi) = -360m_1m_2^2\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)',$$

$$A_8(\psi) = 16m_2^4M_1(\psi) - 648m_1^2m_2\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)',$$

$$A_9(\psi) = 96m_1m_2^3M_1(\psi) - 378m_1^3\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)',$$

$$A_{10}(\psi) = 216m_1^2m_2^2M_1(\psi),$$

$$A_{11}(\psi) = 216m_1^3m_2M_1(\psi),$$

$$A_{12}(\psi) = 81m_1^4M_1(\psi),$$

$$M_1(\psi) = \left[\frac{1}{\gamma'(\psi)}\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)'\right]',$$

$$M_2(\psi) = [\rho - \mu^*\beta^2(\psi)]\frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^*\frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)}.$$

Equation (5.3) is a polynomial of degree 12 in r with coefficients as functions of ψ only. Since r, ψ are independent variables it follows that (5.3) can only hold true for all values of r if the coefficients of different power of r vanish simultaneously and we have

$$A_n(\psi) = 0, \quad n = 0, 3, 4, \ldots, 12. \tag{5.4}$$

Using $m_1 \neq 0$ and $A_{12}(\psi) = 0$ in $A_9(\psi) = 0$, we get

$$\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)' = 0. \tag{5.5}$$

Substituting (5.5) in $A_3(\psi) = 0$, we have

$$\frac{1}{\mu}[\rho - \mu^*\beta^2(\psi)]\frac{1}{\gamma'(\psi)} - 1 = 0. \tag{5.6}$$

Employing (5.5) and (5.6) in $A_0(\psi) = 0$, we obtain

$$3\frac{\gamma''(\psi)}{\gamma'^2(\psi)} + \frac{\mu^*}{\mu}\frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} = 0. \tag{5.7}$$

Upon substitution of (5.5) to (5.7), $A_6(\psi) = 0$ yields

$$\gamma''(\psi) = 0 \tag{5.8}$$

which upon integration with respect to ψ gives

$$\gamma(\psi) = a_1\psi + \psi_0, \tag{5.9}$$

where $a_1 \neq 0$ and ψ_0 are arbitrary constants. Using (5.9) in (5.7) and integrating the resulting equation, we get $\beta(\psi) = \beta_0$, where $\beta_0 \neq 0$ is an arbitrary constant. Employing $\beta(\psi) = \beta_0$ and (5.9) in (5.6), we get $a_1 = (1/\mu)[\rho - \mu^*\beta_0^2]$. Thus, the unknown functions $\gamma(\psi)$ and $\beta(\psi)$ are given by

$$\gamma(\psi) = \frac{1}{\mu}(\rho - \mu^*\beta_0^2)\psi + \psi_0, \qquad \beta(\psi) = \beta_0, \tag{5.10}$$

where ψ_0 and $\beta_0 \neq \sqrt{\rho/\mu^*}$ are arbitrary constants. Using (5.1), (5.10), and $v_1 = (1/r)(\partial\psi/\partial\theta)$, $v_2 = -\partial\psi/\partial r$, we find that the solutions are given by

$$\begin{aligned}
v_1 &= \frac{\mu}{\rho - \mu^*\beta_0^2}\frac{1}{r}, \qquad v_2 = \frac{\mu}{\rho - \mu^*\beta_0^2}[3m_1r^2 + 2m_2r],\\
H_1 &= \frac{\mu\beta_0}{[\rho - \mu^*\beta_0^2]}\left[\frac{\cos\theta}{r} - (3m_1r^2 + 2m_2r)\sin\theta\right],\\
H_2 &= \frac{\mu\beta_0}{[\rho - \mu^*\beta_0^2]}\left[\frac{\sin\theta}{r} + (3m_1r^2 + 2m_2r)\cos\theta\right],\\
p &= \frac{\mu^2}{(\rho - \mu^*\beta_0^2)^2}\left[\frac{9m_1^2}{4}(\rho - 3\mu^*\beta_0^2)r^4 - \frac{\rho}{2r^2} + 2m_2^2(\rho - 2\mu^*\beta_0^2)r^2,\right.\\
&\qquad\left. + 2m_1m_2(2\rho - 5\mu^*\beta_0^2)r^3 - 4m_2(\rho - 2\mu^*\beta_0^2)(\theta - \psi_0)\right] + p_0,\\
\omega &= \frac{\mu}{[\rho - \mu^*\beta_0^2]}[9m_1r + 2m_2], \qquad \Omega = \beta_0\omega,
\end{aligned} \tag{5.11}$$

where p_0 is an arbitrary constant. Since the pressure p must be a single-valued function, we must take $m_2 = 0$.

Finitely conducting flow. Using (4.4) in (4.6) and (4.7), we get

$$\sum_{\substack{n=0\\ n\neq 1,2}}^{12} B_n(\psi)r^n = 0, \tag{5.12}$$

$$\sum_{\substack{n=0\\ n\neq 1}}^{6} C_n(\psi)r^n = 0, \tag{5.13}$$

where

$$
\begin{aligned}
B_0(\psi) &= \frac{2\rho}{\mu}\frac{\gamma''(\psi)}{\gamma'^3(\psi)} + M_1(\psi) + \frac{4\gamma''(\psi)}{\gamma'^2(\psi)},\\
B_3(\psi) &= 9m_1 - 6m_1\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)' - \frac{9m_1\rho}{\mu\gamma'(\psi)},\\
B_4(\psi) &= 8m_2^2M_1(\psi) + 32m_2^2\frac{\gamma''(\psi)}{\gamma'^2(\psi)} - \frac{8m_2^2\rho}{\mu}\frac{\gamma''(\psi)}{\gamma'^3(\psi)},\\
B_5(\psi) &= 24m_1m_2M_1(\psi) + 216m_1m_2\frac{\gamma''(\psi)}{\gamma'^2(\psi)} - \frac{36m_1m_2\rho}{\mu}\frac{\gamma''(\psi)}{\gamma'^3(\psi)},\\
B_6(\psi) &= 18m_1^2 - 72m_2^3\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)' + 279m_1^2\frac{\gamma''(\psi)}{\gamma'^2(\psi)} - \frac{36m_1^2\rho}{\mu}\frac{\gamma''(\psi)}{\gamma'^3(\psi)},\\
B_7(\psi) &= -360m_1m_2^2\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)',\\
B_8(\psi) &= 16m_2^4M_1(\psi) - 648m_1^2m_2\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)',\\
B_9(\psi) &= 96m_1m_2^3M_1(\psi) - 378m_1^3\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)',\\
B_{10}(\psi) &= 216m_1^2m_2^2M_1(\psi),\\
B_{11}(\psi) &= 216m_1^3m_2M_1(\psi),\\
B_{12}(\psi) &= 81m_1^4M_1(\psi),\\
M_1(\psi) &= \left[\frac{1}{\gamma'(\psi)}\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)'\right]',\\
C_0(\psi) &= \frac{\beta(\psi)\gamma''(\psi)}{\gamma'^3(\psi)} - \frac{\beta'(\psi)}{\gamma'^2(\psi)}, \qquad C_2(\psi) = 4m_2\frac{\beta(\psi)}{\gamma'(\psi)} - \Omega_0,\\
C_3(\psi) &= 9m_1\frac{\beta(\psi)}{\gamma'(\psi)}, \qquad C_4(\psi) = 4m_2^2C_0(\psi),\\
C_5(\psi) &= 12m_1m_2C_0(\psi), \qquad C_6(\psi) = 9m_1^2C_0(\psi).
\end{aligned}
\tag{5.14}
$$

Taking $C_3(\psi) = 0$, we obtain

$$\beta(\psi) = 0. \tag{5.15}$$

Thus, we conclude that this streamline pattern is not permissible for a finitely conducting MHD aligned fluid flow.

Non-MHD flow. Employing (4.4) in (4.6), we have

$$\sum_{\substack{n=0,\\ n\neq 1,2}}^{12} D_n(\psi)r^n = 0. \tag{5.16}$$

Equation (5.16) is the same as (5.12) above with $D_n(\psi) = B_n(\psi)$. Using the consequence of $D_{12}(\psi) = 0$ and $D_9(\psi) = 0$ in $D_3(\psi) = 0$, we get

$$9m_1 - \frac{9m_1\rho}{\mu}\frac{1}{\gamma'(\psi)} = 0 \tag{5.17}$$

which implies that $\gamma'(\psi) = \rho/\mu$. Thus, the unknown function $\gamma(\psi)$ is given by

$$\gamma(\psi) = \frac{\rho}{\mu}\psi + \psi_0, \tag{5.18}$$

where ψ_0 is an arbitrary constant of integration. The exact solutions are given by (5.11) with $\beta_0 = 0$. Summing up the above results, we have the following theorem.

Theorem 5.2. *Streamline pattern $\theta - m_1 r^3 - m_2 r^2 =$ constant is not permissible for a finitely conducting MHD aligned flow but is permissible for an infinitely conducting MHD aligned flow with solutions given by (5.11) and non-MHD flows with solutions given by (5.11) with $\beta_0 = 0$.*

Example 5.3 (Flow with $\theta - ar =$ constant as streamlines)**.** We let the family of curves $\theta - ar =$ constant be the streamlines so that we have

$$\theta = ar + \gamma(\psi), \quad \gamma'(\psi) \neq 0, \tag{5.19}$$

where $\gamma(\psi)$ is some unknown function of ψ.

Comparing (5.19) with (4.1), we get

$$f(r) = ar, \qquad g(r) = 1. \tag{5.20}$$

The streamline pattern for this flow is shown in Figure 5.2.

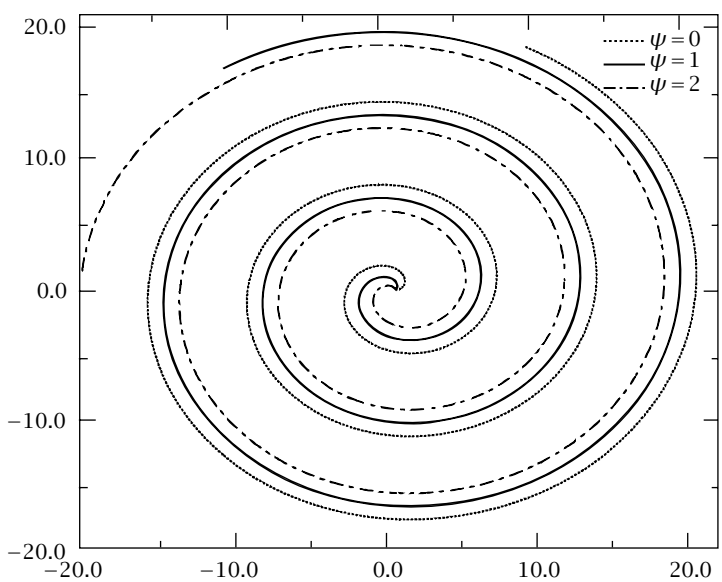

Figure 5.2. Streamline pattern for $\theta - ar = \psi$.

Infinitely conducting flow. Using (4.4) and (4.5) in (4.3), we obtain

$$\sum_{n=0}^{4} A_n(\psi) r^n = 0, \tag{5.21}$$

where

$$\begin{aligned}
A_0(\psi) &= \frac{4\gamma''(\psi)}{\gamma'^2(\psi)} + \left[\frac{1}{\gamma'(\psi)}\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)'\right]' + \frac{2}{\mu}[\rho - \mu^*\beta^2(\psi)]\frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \frac{2\mu}{\mu^*}\frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)},\\
A_1(\psi) &= a + 2a\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)' + \frac{a}{\mu}[\rho - \mu^*\beta^2(\psi)]\frac{1}{\gamma'(\psi)},\\
A_2(\psi) &= -a^2\frac{\gamma''(\psi)}{\gamma'^2(\psi)} + 2a^2\left[\frac{1}{\gamma'(\psi)}\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)'\right]',\\
A_3(\psi) &= -2a^3\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)', \qquad A_4(\psi) = a^4\left[\frac{1}{\gamma'(\psi)}\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)'\right]'.
\end{aligned} \tag{5.22}$$

Equation (5.21) is a fourth degree polynomial in r with coefficients as functions of ψ only. Since r, ψ are independent variables it follows that (5.21) can only hold true for all values of r if the coefficients of different powers of r vanish simultaneously and we have

$$A_0(\psi) = A_1(\psi) = A_2(\psi) = A_3(\psi) = A_4(\psi) = 0. \tag{5.23}$$

Equation $A_3(\psi) = 0$ holds true in one of the following three cases:

(i) $a \neq 0$, $(\gamma''(\psi)/\gamma'^3(\psi))' = 0$,

(ii) $a = 0$, $(\gamma''(\psi)/\gamma'^3(\psi))' \neq 0$,

(iii) $a = 0$, $(\gamma''(\psi)/\gamma'^3(\psi))' = 0$.

In the following, we study these three cases separately.

Case (i). In this case, all the coefficients $A_n(\psi)$, $n = 0, 1, \ldots, 4$ vanish simultaneously if

$$\gamma(\psi) = -\frac{1}{\mu}(\rho - \mu^*\beta_0^2)\psi + \psi_0, \qquad \beta(\psi) = \beta_0, \tag{5.24}$$

where ψ_0 and $\beta_0 \neq \sqrt{\rho/\mu^*}$ are arbitrary constants.

Using (5.24) in (5.19), we obtain

$$\theta = ar - \frac{1}{\mu}(\rho - \mu^*\beta_0^2)\psi + \psi_0, \tag{5.25}$$

and the solutions for this flow are given by

$$\begin{aligned}
v_1 &= -\frac{\mu}{[\rho - \mu^*\beta_0^2]r}, \qquad v_2 = -\frac{a\mu}{[\rho - \mu^*\beta_0^2]},\\
H_1 &= \frac{\mu\beta_0}{[\rho - \mu^*\beta_0^2]}\left[a\sin\theta - \frac{\cos\theta}{r}\right], \qquad H_2 = -\frac{\mu\beta_0}{[\rho - \mu^*\beta_0^2]}\left[a\cos\theta + \frac{\sin\theta}{r}\right],\\
p &= \frac{a^2\mu^2}{(\rho - \mu^*\beta_0^2)}\ln r - \frac{\rho\mu^2}{2(\rho - \mu^*\beta_0^2)^2 r^2} + p_0, \qquad \omega = -\frac{a\mu}{[\rho - \mu^*\beta_0^2]r}, \qquad \Omega = \beta_0\omega,
\end{aligned} \tag{5.26}$$

where p_0 is an arbitrary constant. Thus, we have: $\theta - ar =$ constant is a permissible streamline pattern for an infinitely conducting MHD aligned fluid flow and the exact integral for this flow is given by (5.26).

Case (ii). In this case, $A_n(\psi) = 0$, $n = 0, 1, \ldots, 4$, are identically satisfied if

$$\beta^2(\psi) = \frac{1}{\mu^*}\left[\rho - \mu\gamma'(\psi)\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)' + 4\mu\gamma'(\psi) + B\mu\gamma'^2(\psi)\right], \tag{5.27}$$

where B is an arbitrary constant of integration. Equation (5.27) is one equation in two unknowns $\beta(\psi)$ and $\gamma(\psi)$ when $(\gamma''(\psi)/\gamma'^3(\psi))' \neq 0$. There are two ways of getting solutions for this case. One way is to prescribe $\beta(\psi)$ and solve (5.27) to get $\gamma(\psi)$ and the second way is to choose a $\gamma(\psi)$ such that $(\gamma''(\psi)/\gamma'^3(\psi))' \neq 0$ and use (5.27) to find $\beta(\psi)$.

The exact solutions, for this flow, are given by

$$\begin{aligned}
v_1 &= \frac{1}{r\gamma'(\psi)}, \qquad v_2 = 0,\\
H_1 &= \frac{\cos\theta}{r}\frac{\beta(\psi)}{\gamma'(\psi)}, \qquad H_2 = \frac{\sin\theta}{r}\frac{\beta(\psi)}{\gamma'(\psi)},\\
p &= \frac{1}{2r^2}\left[\frac{\mu}{\gamma'(\psi)}\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)' - \frac{\rho}{\gamma'^2(\psi)}\right] + p_0,\\
\omega &= \frac{1}{r^2}\frac{\gamma''(\psi)}{\gamma'^3(\psi)}, \qquad \Omega = \beta(\psi)\omega - \frac{1}{r^2}\frac{\beta'(\psi)}{\gamma'^2(\psi)},
\end{aligned} \tag{5.28}$$

where p_0 is an arbitrary constant and $\beta(\psi), \gamma(\psi)$ are arbitrary functions of ψ such that (5.27) is satisfied.

Thus, we have: $\theta =$ constant is a permissible streamline pattern for steady plane rotational infinitely conducting MHD aligned fluid flow and the exact integral for this flow is given by (5.28), where $\beta(\psi)$ and $\gamma(\psi)$ are arbitrary functions of ψ such that (5.27) and $(\gamma''(\psi)/\gamma'^3(\psi))' \neq 0$ are satisfied.

As an example, we take $\gamma(\psi) = 6\mu/\rho\psi$. With this choice, equation (5.27) gives

$$\beta^2(\psi) = -\frac{1}{\mu^*}\left[\frac{24\mu^2}{\rho\psi^2} + \frac{36\mu^3 B}{\rho^2\psi^4}\right], \tag{5.29}$$

and the solutions (5.28) take the form

$$\begin{aligned}
v_1 &= -\frac{6\mu}{\rho r\theta^2}, \qquad v_2 = 0,\\
H_1 &= -\frac{6\mu}{\rho r}\beta(\psi)\frac{\cos\theta}{\theta^2}, \qquad H_2 = -\frac{6\mu}{\rho r}\beta(\psi)\frac{\sin\theta}{\theta^2},\\
p &= p_0, \qquad \omega = -\frac{12\mu}{\rho r^2\theta^3}, \qquad \Omega = \beta(\psi)\omega - \frac{9\mu^2\beta'(\psi)}{\rho^2 r^2\theta^4},
\end{aligned} \tag{5.30}$$

where $\beta(\psi)$ is given by (5.29).

Case (iii). Integrating $(\gamma''(\psi)/\gamma'^3(\psi))' = 0$ three times with respect to ψ, we find that the function $\gamma(\psi)$ is given implicitly by

$$c_1\gamma^2(\psi) + c_2\gamma(\psi) + c_3 = \psi, \tag{5.31}$$

where c_1, c_2, and c_3 are arbitrary constants such that c_1 and c_2 are not zero simultaneously. Using (5.31) and $a = 0$ in $A_n(\psi) = 0$, $n = 0, 1, \ldots, 4$, we obtain

$$\beta^2(\psi) = \frac{1}{\mu^*}\left[\rho + 4\mu\gamma'(\psi) - \mu B\gamma'^2(\psi)\right], \tag{5.32}$$

where $\gamma(\psi)$ is given by (5.31) and B is an arbitrary constant of integration.

The exact solutions for this flow are given by

$$\begin{aligned}
v_1 &= \frac{1}{r}\left[2c_1\theta + c_2\right], \qquad v_2 = 0,\\
H_1 &= \frac{\beta(\psi)}{r}(2c_1\theta + c_2)\cos\theta, \qquad H_2 = \frac{\beta(\psi)}{r}(2c_1\theta + c_2)\sin\theta,\\
p &= p_0 - \frac{\rho}{2r^2}\left[2c_1\theta + c_2\right]^2,\\
\omega &= -\frac{2c_1}{r^2}, \qquad \Omega = \beta(\psi)\omega - \frac{\beta'(\psi)}{r^2}(2c_1\theta + c_2)^2,
\end{aligned} \tag{5.33}$$

where p_0 is an arbitrary constant, $\gamma(\psi)$ and $\beta(\psi)$ are given by (5.31) and (5.32) respectively. Since the pressure must be single-valued, we must take $c_1 = 0$. If $c_1 = 0$, then the flow turns out to be irrotational.

Finitely conducting flow. Using (4.4) in (4.6) and (4.7), we get

$$\sum_{n=0}^{4} B_n(\psi) r^n = 0, \qquad \sum_{n=0}^{2} C_n(\psi) r^n = 0, \tag{5.34}$$

where

$$\begin{aligned}
B_0(\psi) &= \frac{4\gamma''(\psi)}{\gamma'^2(\psi)} + \left[\frac{1}{\gamma'(\psi)}\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)'\right]' + \frac{2\rho}{\mu}\frac{\gamma''(\psi)}{\gamma'^3(\psi)},\\
B_1(\psi) &= a + 2a\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)' + \frac{a\rho}{\mu}\frac{1}{\gamma'(\psi)},\\
B_2(\psi) &= -a^2\frac{\gamma''(\psi)}{\gamma'^2(\psi)} + 2a^2\left[\frac{1}{\gamma'(\psi)}\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)'\right]',\\
B_3(\psi) &= -2a^3\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)',\\
B_4(\psi) &= a^4\left[\frac{1}{\gamma'(\psi)}\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)'\right]',\\
C_0(\psi) &= \frac{\beta(\psi)\gamma''(\psi)}{\gamma'^3(\psi)} - \frac{\beta'(\psi)}{\gamma'^2(\psi)},\\
C_1(\psi) &= \frac{a\beta(\psi)}{\gamma'(\psi)}, \qquad C_2(\psi) = a^2\frac{\beta(\psi)\gamma''(\psi)}{\gamma'^3(\psi)} - \frac{a^2\beta'(\psi)}{\gamma'^2(\psi)} - \Omega_0.
\end{aligned} \tag{5.35}$$

Equations (5.34) must hold true for all values of r. Since r, ψ are independent variables, then we have

$$B_0(\psi) = B_1(\psi) = B_2(\psi) = B_3(\psi) = B_4(\psi) = C_0(\psi) = C_1(\psi) = C_2(\psi) = 0. \tag{5.36}$$

Requiring $C_1(\psi) = 0$, we have

$$a = 0 \tag{5.37}$$

since $\beta(\psi) \neq 0$. Using $a = 0$, the equations $B_1(\psi) = 0$, $B_2(\psi) = 0$, $B_3(\psi) = 0$, $B_4(\psi) = 0$ are identically satisfied and $B_0(\psi) = 0$, $C_0(\psi) = 0$, $C_2(\psi) = 0$ give

$$\begin{gathered}\frac{1}{\gamma'(\psi)}\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)' - \frac{4}{\gamma'(\psi)} - \frac{\rho}{\mu}\frac{1}{\gamma'^2(\psi)} = \psi_0,\\ \beta^2(\psi) = \beta_0\gamma'^2(\psi), \qquad \Omega_0 = 0,\end{gathered} \tag{5.38}$$

where ψ_0 and β_0 are arbitrary constants integration. Proceeding as in infinitely conducting flow, we find that the exact solutions for this flow are given by

$$\begin{aligned} v_1 &= \frac{1}{r\gamma'(\psi)}, \qquad v_2 = 0,\\ H_1 &= \frac{\cos\theta}{r}\frac{\beta(\psi)}{\gamma'(\psi)}, \qquad H_2 = \frac{\sin\theta}{r}\frac{\beta(\psi)}{\gamma'(\psi)},\\ p &= \frac{1}{2r^2}\left[\frac{\mu}{\gamma'(\psi)}\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)' - \frac{\rho}{\gamma'^2(\psi)}\right] + p_0, \qquad \omega = \frac{1}{r^2}\frac{\gamma''(\psi)}{\gamma'^3(\psi)},\end{aligned} \tag{5.39}$$

where p_0 is an arbitrary constant and $\gamma(\psi), \beta(\psi)$ are given by (5.38). Thus, we have: $\theta =$ constant is a permissible streamline pattern for a finitely conducting MHD aligned fluid flow and the exact solutions of this flow are given by (5.39) with $\gamma(\psi)$ and $\beta(\psi)$ given by (5.38).

Non-MHD flow. Employing (4.4) in (4.6), we obtain

$$\sum_{n=0}^{4} D_n(\psi) r^n = 0, \tag{5.40}$$

where

$$\begin{aligned} D_0(\psi) &= \frac{4\gamma''(\psi)}{\gamma'^2(\psi)} + \left[\frac{1}{\gamma'(\psi)}\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)'\right]' + \frac{2\rho}{\mu}\frac{\gamma''(\psi)}{\gamma'^3(\psi)},\\ D_1(\psi) &= a + 2a\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)' + \frac{a\rho}{\mu}\frac{1}{\gamma'(\psi)},\\ D_2(\psi) &= -a^2\frac{\gamma''(\psi)}{\gamma'^2(\psi)} + 2a^2\left[\frac{1}{\gamma'(\psi)}\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)'\right]',\\ D_3(\psi) &= -2a^3\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)', \qquad D_4(\psi) = a^4\left[\frac{1}{\gamma'(\psi)}\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)'\right]'.\end{aligned} \tag{5.41}$$

Since r and ψ are independent variables and (5.40) is a fourth-degree polynomial with coefficients as functions of ψ only, then we must have

$$D_0(\psi) = D_1(\psi) = D_2(\psi) = D_3(\psi) = D_4(\psi) = 0. \tag{5.42}$$

Requiring $D_3(\psi) = 0$, we get the following three cases:

(i) $a \neq 0$, $(\gamma''(\psi)/\gamma'^3(\psi))' = 0$,

(ii) $a = 0$, $(\gamma''(\psi)/\gamma'^3(\psi))' \neq 0$,

(iii) $a = 0$, $(\gamma''(\psi)/\gamma'^3(\psi))' = 0$.

We study these three cases separately as follows.

Case (i). Using $(\gamma''(\psi)/\gamma'^3(\psi))' = 0$, in $D_0(\psi) = 0$, $D_1(\psi) = 0$, $D_2(\psi) = 0$, and in $D_4(\psi) = 0$, we have

$$\gamma(\psi) = -\frac{\rho}{\mu}\psi + \psi_0, \tag{5.43}$$

where ψ_0 is an arbitrary constant. In this case, solutions are given by (5.26) with $\beta_0 = 0$.

Case (ii). With $a = 0$, equations $D_1(\psi) = 0$, $D_2(\psi) = 0$, and $D_4(\psi) = 0$ are identically satisfied and $D_0(\psi) = 0$ gives

$$\frac{1}{\gamma'(\psi)}\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)' - \frac{4}{\gamma'(\psi)} - \frac{\rho}{\mu}\frac{1}{\gamma'^2(\psi)} = \psi_0, \tag{5.44}$$

where ψ_0 is an arbitrary constant. For this case, exact solutions are given by (5.28) with $\beta(\psi) = 0$.

Case (iii). For this case, equations (5.42) are identically satisfied if

$$\gamma(\psi) = c_1\psi + c_2, \tag{5.45}$$

where $c_1 \neq 0$ and c_2 are arbitrary constants. The solutions for this case are given by

$$v_1 = \frac{1}{c_1 r}, \qquad v_2 = 0, \qquad p = p_0 - \frac{\rho}{2c_1^2 r^2}, \qquad \omega = 0, \tag{5.46}$$

where p_0 is an arbitrary constant.

Summing up, we have the following theorem.

Theorem 5.4. *Streamline pattern $\theta - ar$ = constant in a steady plane motion is permissible for an infinitely conducting MHD aligned and non-MHD fluid flow. It is also permissible for a finitely conducting MHD aligned flow if $a = 0$.*

Example 5.5 (Flow with $\theta - a_1 r^m - a_2 \ln r$ = constant as streamlines)**.** We assume

$$\theta = a_1 r^m + a_2 \ln r + \gamma(\psi); \quad \gamma'(\psi) \neq 0, \tag{5.47}$$

where a_1, a_2, and m are arbitrary constants and, therefore, we have

$$f(r) = a_1 r^m + a_2 \ln r, \qquad g(r) = 1. \tag{5.48}$$

The streamline pattern for this flow is shown in Figure 5.3.

Infinitely conducting flow. Proceeding as in Example 5.5, we find that the functions $\gamma(\psi)$ and $\beta(\psi)$ must satisfy

$$\sum_{n=0}^{4} A_n(\psi) r^{nm-3} = 0, \tag{5.49}$$

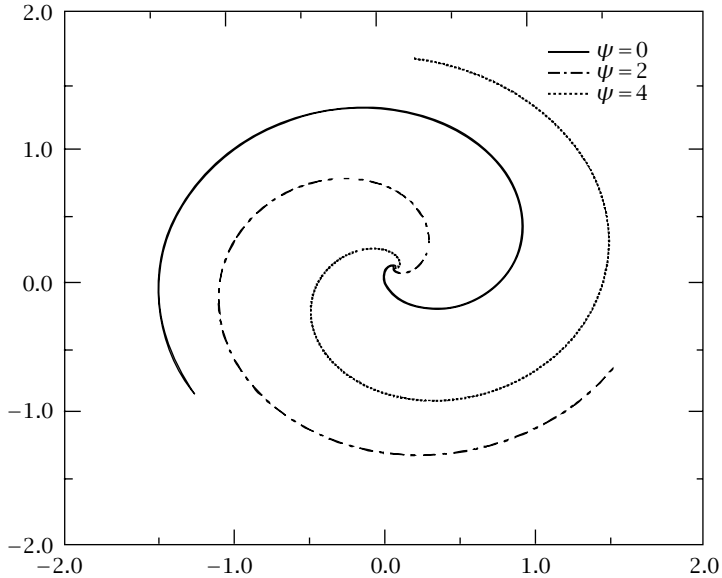

FIGURE 5.3. Streamline pattern for $\theta - a_1 r^m - a_2 \ln r = \psi$, $m = 3$.

where

$$
\begin{aligned}
A_0(\psi) &= (1+a_2^2)^2\left[\frac{1}{\gamma'(\psi)}\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)'\right]' + 4a_2(1+a_2^2)\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)' \\
&\quad + 4(1+a_2^2)\frac{\gamma''(\psi)}{\gamma'^2(\psi)} + \frac{2}{\mu}(1+a_2^2)\left[(\rho-\mu^*\beta^2(\psi))\frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^*\frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)}\right], \\
A_1(\psi) &= 4ma_1a_2(1+a_2^2)\left[\frac{1}{\gamma'(\psi)}\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)'\right]' + a_1a_2m(2m^2-7m+10)\frac{\gamma''(\psi)}{\gamma'^2(\psi)} \\
&\quad - 2a_1m(m-2)(1+3a_2^2)\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)' - \frac{\mu^*}{\mu}a_1a_2m(m-2)\frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \\
&\quad - \frac{1}{\mu}a_1m(m-2)[\rho-\mu^*\beta^2(\psi)]\left[\frac{m}{\gamma'(\psi)} + a_2\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right] + a_1m^2(m-2)^2, \\
A_2(\psi) &= 2a_1^2m^2(1+3a_2^2)\left[\frac{1}{\gamma'(\psi)}\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)'\right]' \\
&\quad - 12a_1^2a_2m^2(m-1)\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)' + a_1^2m^2(5m^2-6m+4)\frac{\gamma''(\psi)}{\gamma'^2(\psi)} \\
&\quad - \frac{2}{\mu}a_1^2m^2(m-1)\left[(\rho-\mu^*\beta^2(\psi))\frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^*\frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)}\right], \\
A_3(\psi) &= 4a_1^3a_2m^3\left[\frac{1}{\gamma'(\psi)}\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)'\right]' - 2a_1^3m^3(3m-2)\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)', \\
A_4(\psi) &= a_1^2m^4\left[\frac{1}{\gamma'(\psi)}\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)'\right]'.
\end{aligned}
\tag{5.50}
$$

Equation (5.49) holds true for all values of r provided all coefficients vanish simultaneously and we have

$$A_0(\psi) = A_1(\psi) = A_2(\psi) = A_3(\psi) = A_4(\psi) = 0. \tag{5.51}$$

In particular, $A_1(\psi) = 0$, $A_2(\psi) = 0$, $A_3(\psi) = 0$, and $A_4(\psi) = 0$ are identically satisfied in one of the following four cases:

(a) $a_1 \neq 0$, $m \neq 2$, $\gamma'(\psi) = (1/\mu(m-2))(\rho - \mu^*\beta_0^2)$, $\beta(\psi) = \beta_0$, $\beta_0 \neq \sqrt{\rho/\mu^*}$,

(b) $\gamma''(\psi) = 0$, $a_1 \neq 0$, $m = 2$, $\beta(\psi) = \beta_0$,

(c) $a_1 = 0$, $[(1/\gamma'(\psi))(\gamma''(\psi)/\gamma'^3(\psi))']' \neq 0$,

(d) $a_1 = 0$, $[(1/\gamma'(\psi))(\gamma''(\psi)/\gamma'^3(\psi))']' = 0$.

We study these four cases separately as follows.

Case (a). In this case, $A_0(\psi) = 0$ is identically satisfied and we have: $\theta - a_1 r^m - a_2 \ln r =$ constant with $m \neq 2$ is a permissible streamline pattern and the exact integral for this flow is given by

$$\begin{aligned}
v_1 &= \frac{\mu(m-2)}{r[\rho - \mu^*\beta_0^2]}, \qquad v_2 = \frac{\mu(m-2)}{\rho - \mu^*\beta_0^2}\left[m a_1 r^{m-1} + \frac{a_2}{r}\right],\\
H_1 &= \frac{\mu\beta_0(m-2)}{\rho - \mu\beta_0^2}\left[\frac{\cos\theta - a_2\sin\theta}{r} - m a_1 r^{m-1}\sin\theta\right],\\
H_2 &= \frac{\mu\beta_0(m-2)}{\rho - \mu\beta_0^2}\left[\frac{\sin\theta + a_2\cos\theta}{r} + m a_1 r^{m-1}\cos\theta\right],\\
p &= \begin{cases}
\dfrac{\mu^2 a_1 m^2(m-2)^2}{\rho - \mu^*\beta_0^2}\left[\dfrac{a_1 m}{2(m-1)} r^{2m-2} + \dfrac{a_2}{m-2} r^{m-2}\right]\\
\qquad -\dfrac{1}{2}\rho\dfrac{\mu^2(m-2)^2}{(\rho - \mu^*\beta_0^2)^2}\left(\dfrac{1+a_2^2}{r^2} + m^2 a_1^2 r^{2m-2} + 2m a_1 a_2 r^{m-2}\right) + p_0; \quad m \neq 1,\\
\dfrac{\mu^2 a_1}{\rho - \mu^*\beta_0^2}\left[a_1 \ln r - \dfrac{a_2}{r}\right] - \dfrac{1}{2}\rho\dfrac{\mu^2}{(\rho - \mu^*\beta_0^2)^2}\left(\dfrac{1+a_2^2}{r^2} + \dfrac{2a_1 a_2}{r}\right) + p_0; \quad m = 1,
\end{cases}\\
\omega &= \frac{\mu m^2(m-2)a_1}{\rho - \mu\beta_0^2} r^{m-2}, \qquad \Omega = \beta_0\omega,
\end{aligned} \tag{5.52}$$

where p_0 is an arbitrary constant of integration and $m \neq 2$.

Case (b). Since $\gamma''(\psi) = 0$, then we get

$$\gamma(\psi) = b_1\psi + b_2, \tag{5.53}$$

where $b_1 \neq 0$ and b_2 are arbitrary constants. Using $\beta(\psi) = \beta_0$ and (5.53), $A_0(\psi)$ is identically satisfied and we have: $\theta - a_1 r^2 - a_2 \ln r =$ constant is a possible streamline pattern and the exact integral associated with this flow is given by

$$
\begin{aligned}
v_1 &= \frac{1}{b_1 r}, \qquad v_2 = \frac{1}{b_1}\left[2a_1 r + \frac{a_2}{r}\right],\\
H_1 &= \frac{\beta_0}{b_1}\left[\frac{\cos\theta - a_2\sin\theta}{r} - 2a_1 r\sin\theta\right],\\
H_2 &= \frac{\beta_0}{b_1}\left[\frac{\sin\theta + a_2\cos\theta}{r} + 2a_1 r\cos\theta\right],\\
p &= p_0 - \frac{4a_1}{b_1^2}(\rho - \mu^*\beta_0^2)[\theta - a_1 r^2 - a_2\ln r - b_2] - \frac{1}{2}\frac{\rho}{b_1^2}\left[\frac{1+a_2^2}{r^2} + 4a_1^2 r^2 + 4a_1 a_2\right],\\
\omega &= \frac{4a_1}{b_1}, \qquad \Omega = \beta_0\omega,
\end{aligned}
\tag{5.54}
$$

where p_0 is an arbitrary constant. Since the pressure function p must be single-valued, we must take $\beta_0^2 = \rho/\mu^*$.

Case (c). All coefficients $A_n(\psi)$, $n = 0,1,\dots,4$ vanish simultaneously if

$$
\beta^2(\psi) = \frac{\mu}{\mu^*}\left(4\gamma'(\psi) - (1+a_2^2)\gamma'(\psi)\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)' - 4a_2\frac{\gamma''(\psi)}{\gamma'(\psi)} - b_3\gamma'^2(\psi) + \frac{\rho}{\mu}\right),
\tag{5.55}
$$

where b_3 is an arbitrary constant and $\gamma(\psi)$ is an arbitrary function of ψ such that $[(1/\gamma'(\psi))(\gamma''(\psi)/\gamma'^3(\psi))']' \neq 0$. Thus, $\theta - a_2\ln r =$ constant can serve as streamline pattern and the exact integral for this rotational flow is given by

$$
\begin{aligned}
v_1 &= \frac{1}{r\gamma'(\psi)}, \qquad v_2 = \frac{a_2}{r\gamma'(\psi)},\\
H_1 &= \frac{\cos\theta - a_2\sin\theta}{r}\frac{\beta(\psi)}{\gamma'(\psi)}, \qquad H_2 = \frac{\sin\theta + a_2\cos\theta}{r}\frac{\beta(\psi)}{\gamma'(\psi)},\\
p &= \frac{\mu}{2}(1+a_2^2)\frac{1}{r^2}\left[2a_2\frac{\gamma''(\psi)}{\gamma'^3(\psi)} + (1+a_2^2)\frac{1}{\gamma'(\psi)}\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)'\right] - \frac{1}{2}\frac{\rho}{\gamma'^2(\psi)}\frac{1+a_2^2}{r^2} + p_0,\\
\omega &= \frac{1+a_2^2}{r^2}\frac{\gamma''(\psi)}{\gamma'^3(\psi)}, \qquad \Omega = \beta(\psi)\omega - \frac{1+a_2^2}{r^2}\frac{\beta'(\psi)}{\gamma'^2(\psi)},
\end{aligned}
\tag{5.56}
$$

where p_0 is an arbitrary constant, $\beta(\psi)$ is given by (5.55) and $\gamma(\psi)$ is an arbitrary function of ψ.

Case (d). Integrating $[(1/\gamma'(\psi))(\gamma''(\psi)/\gamma'^3(\psi))']' = 0$ four times with respect to ψ, we obtain

$$
c_1\gamma^3(\psi) + c_2\gamma^2(\psi) + c_3\gamma(\psi) + c_4 = \psi,
\tag{5.57}
$$

where c_1, c_2, c_3, and c_4 are arbitrary constants of integration such that c_1, c_2, and c_3 are not zero simultaneously. Using $a_1 = 0$ and $[(1/\gamma'(\psi))(\gamma''(\psi)/\gamma'^3(\psi))']' = 0$ in $A_0(\psi) = 0$ and integrating the resulting equation with respect to ψ, we get

$$\beta^2(\psi) = \frac{\mu}{\mu^*}\left[4\gamma'(\psi) - 4a_2\frac{\gamma''(\psi)}{\gamma'(\psi)} + \frac{\rho}{\mu} + c_5\gamma'^2(\psi)\right], \tag{5.58}$$

where c_5 is an arbitrary constant of integration and $\gamma(\psi)$ is given implicitly by (5.57).

The exact solutions for this flow are given by

$$\begin{aligned}
v_1 &= \frac{1}{r}[3c_1(\theta - a_2\ln r)^2 + 2c_2(\theta - a_2\ln r) + c_3],\\
v_2 &= \frac{a_2}{r}[3c_1(\theta - a_2\ln r)^2 + 2c_2(\theta - a_2\ln r) + c_3],\\
H_1 &= \frac{\cos\theta - a_2\sin\theta}{r}\beta(\psi)[3c_1(\theta - a_2\ln r)^2 + 2c_2(\theta - a_2\ln r) + c_3],\\
H_2 &= \frac{\sin\theta + a_2\cos\theta}{r}\beta(\psi)[3c_1(\theta - a_2\ln r)^2 + 2c_2(\theta - a_2\ln r) + c_3],\\
p &= (1+a_2^2)\frac{1}{r^2}(-\mu a_2[6c_1(\theta - a_2\ln r) + 2c_2] - 3\mu(1+a_2^2)c_1\\
&\qquad - \rho[9c_1^2(\theta - a_2\ln r)^4 + 12c_1c_2(\theta - a_2\ln r)^3 + 4c_2c_3(\theta - a_2\ln r)\\
&\qquad + (6c_1c_3 + 4c_2^2)(\theta - a_2\ln r)^2 + c_3^2]) + p_0,\\
\omega &= -\frac{1+a_2^2}{r^2}[6c_2(\theta - a_2\ln r) + 2c_2],\\
\Omega &= \beta(\psi)\omega - \frac{1+a_2^2}{r^2}\beta'(\psi)[3c_1(\theta - a_2\ln r)^2 + 2c_2(\theta - a_2\ln r) + c_3]^2,
\end{aligned} \tag{5.59}$$

where p_0 is an arbitrary constant of integration and $\beta(\psi)$ is given by (5.58). Since the pressure function must be single-valued, we must take $c_1 = c_2 = 0$. If $c_1 = c_2 = 0$, then $\omega = 0$ and the flow is irrotational.

Finitely conducting flow. Employing (4.4) in (4.6) and (4.7), we get

$$\sum_{n=0}^{4} B_n(\psi)r^{nm-3} = 0, \qquad \sum_{n=0}^{2} C_n(\psi)r^{nm-2} + C_3(\psi) = 0, \tag{5.60}$$

where

$$\begin{aligned}
B_0(\psi) &= (1+a_2^2)^2\left[\frac{1}{\gamma'(\psi)}\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)'\right]' + 4a_2(1+a_2^2)\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)'\\
&\quad + 4(1+a_2^2)\frac{\gamma''(\psi)}{\gamma'^2(\psi)} + \frac{2\rho}{\mu}\frac{\gamma''(\psi)}{\gamma'^3(\psi)},\\
B_1(\psi) &= 4ma_1a_2(1+a_2^2)\left[\frac{1}{\gamma'(\psi)}\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)'\right]' + a_1a_2m(2m^2 - 7m + 10)\frac{\gamma''(\psi)}{\gamma'^2(\psi)}\\
&\quad - 2a_1m(m-2)(1+3a_2^2)\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)' + a_1m^2(m-2)^2\\
&\quad - \frac{a_1m\rho}{\mu}(m-2)\left[\frac{m}{\gamma'(\psi)} + a_2\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right],
\end{aligned}$$

$$
\begin{aligned}
B_2(\psi) &= 2a_1^2m^2(1+3a_2^2)\left[\frac{1}{\gamma'(\psi)}\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)'\right]' - 12a_1^2a_2m^2(m-1)\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)' \\
&\quad + a_1^2m^2(5m^2-6m+4)\frac{\gamma''(\psi)}{\gamma'^2(\psi)} - \frac{2\rho}{\mu}a_1^2m^2(m-1)\frac{\gamma''(\psi)}{\gamma'^3(\psi)}, \\
B_3(\psi) &= 4a_1^3a_2m^3\left[\frac{1}{\gamma'(\psi)}\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)'\right]' - 2a_1^3m^3(3m-2)\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)', \\
B_4(\psi) &= a_1^2m^4\left[\frac{1}{\gamma'(\psi)}\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)'\right]', \\
C_0(\psi) &= (1+a_2^2)\left[\frac{\beta(\psi)\gamma''(\psi)}{\gamma'^3(\psi)} - \frac{\beta'(\psi)}{\gamma'^2(\psi)}\right], \\
C_1(\psi) &= a_1m^2\frac{\beta(\psi)}{\gamma'(\psi)} + 2a_1a_2m\left[\frac{\beta(\psi)\gamma''(\psi)}{\gamma'^3(\psi)} - \frac{\beta'(\psi)}{\gamma'^2(\psi)}\right], \\
C_2(\psi) &= a_1^2m^2\left[\frac{\beta(\psi)\gamma''(\psi)}{\gamma'^3(\psi)} - \frac{\beta'(\psi)}{\gamma'^2(\psi)}\right], \qquad C_3(\psi) = -\Omega_0.
\end{aligned}
\tag{5.61}
$$

Requiring (5.60) to hold true for all values of r, we get

$$B_i(\psi) = C_j(\psi) = 0, \quad i = 0,1,2,3,4,\ j = 0,1,2,3. \tag{5.62}$$

From $C_3(\psi) = 0$ and $C_0(\psi) = 0$, we have

$$\Omega_0 = 0 \qquad \text{and} \qquad \beta(\psi) = k_1\gamma'(\psi) + k_2, \tag{5.63}$$

respectively, where k_1 and k_2 are arbitrary constants of integration. Using (5.63) in $C_1(\psi) = 0$, we obtain

$$a_1 = 0. \tag{5.64}$$

Employing (5.64), we find that $B_i(\psi) = 0$, $i = 0,1,2,3,4$ are identically satisfied if

$$(1+a_2^2)\left[\frac{1}{\gamma'(\psi)}\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)'\right]' + 4a_2\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)' + 4\frac{\gamma''(\psi)}{\gamma'^2(\psi)} + \frac{2\rho}{\mu}\frac{\gamma''(\psi)}{\gamma'^3(\psi)} = 0. \tag{5.65}$$

Thus, the exact solutions for finitely conducting flow are given by

$$
\begin{aligned}
v_1 &= \frac{1}{r\gamma'(\psi)}, \qquad v_2 = \frac{a_2}{r\gamma'(\psi)}, \\
H_1 &= \frac{\cos\theta - a_2\sin\theta}{r}\frac{\beta(\psi)}{\gamma'(\psi)}, \qquad H_2 = \frac{\sin\theta + a_2\cos\theta}{r}\frac{\beta(\psi)}{\gamma'(\psi)}, \\
p &= \frac{\mu}{2r^2}(1+a_2^2)\left[2a_2\frac{\gamma''(\psi)}{\gamma'^3(\psi)} + (1+a_2^2)\frac{1}{\gamma'(\psi)}\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)'\right] - \frac{1}{2}\rho\frac{1+a_2^2}{r^2\gamma'^2(\psi)} + p_0, \\
\omega &= \frac{1+a_2^2}{r^2}\frac{\gamma''(\psi)}{\gamma'^3(\psi)}, \qquad \Omega = \beta(\psi)\omega - \frac{1+a_2^2}{r^2}\frac{\beta'(\psi)}{\gamma'^2(\psi)},
\end{aligned}
\tag{5.66}
$$

where p_0 is an arbitrary constant, $\beta(\psi)$ is given by (5.63) and $\gamma(\psi)$ is given by (5.65). Since the pressure function should be single-valued, the function $\gamma(\psi)$ must also

satisfy the following equation:

$$2\mu a_2\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)' + \mu(1+a_2^2)\left[\frac{1}{\gamma'(\psi)}\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)'\right]' + \frac{2\rho\gamma''(\psi)}{\gamma'^3(\psi)} = 0. \tag{5.67}$$

Using this equation and (5.65), we find that $\gamma''(\psi) = 0$ and the flow is irrotational.

Non-MHD flow. Substituting (4.4) in (4.6), we obtain

$$\sum_{n=0}^{4} D_n(\psi) r^{nm-3} = 0, \tag{5.68}$$

where $D_n(\psi)$ are given by $B_n(\psi)$ of the finitely conducting case. The coefficients $D_n(\psi)$, $n = 1,\dots,4$ vanish simultaneously in one of the following four cases:

(a) $a_1 \neq 0$, $m \neq 2$, $\gamma'(\psi) = \rho/\mu(m-2)$,

(b) $a_1 \neq 0$, $m = 2$, $\gamma''(\psi) = 0$,

(c) $a_1 = 0$, $[(1/\gamma'(\psi))(\gamma''(\psi)/\gamma'^3(\psi))']' \neq 0$,

(d) $a_1 = 0$, $[(1/\gamma'(\psi))(\gamma''(\psi)/\gamma'^3(\psi))']' = 0$.

We study these four cases separately as follows:

Case (a). In this case $D_0(\psi) = 0$ is also satisfied and the exact integral is given by (5.52) with $\beta_0 = 0$.

Case (b). Since $\gamma''(\psi) = 0$, we get

$$\gamma(\psi) = b_1\psi + b_2, \tag{5.69}$$

where $b_1 \neq 0$ and b_2 are arbitrary constants. Using $\gamma''(\psi) = 0, D_0(\psi) = 0$ is identically satisfied and the exact solutions are given by (5.54) with $\beta_0 = 0$.

Case (c). All coefficients $D_n(\psi)$, $n = 0,1,\dots,4$ vanish simultaneously if

$$(1+a_2^2)\left[\frac{1}{\gamma'(\psi)}\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)'\right]' + 4a_2\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)' + 4\frac{\gamma''(\psi)}{\gamma'^2 v(\psi)} + \frac{2\rho}{\mu}\frac{\gamma''(\psi)}{\gamma'^3(\psi)} = 0, \tag{5.70}$$

and the exact solutions are given by (5.56) with $\beta(\psi) = 0$ and $\gamma(\psi)$ given by (5.70).

Case (d). Integrating $[(1/\gamma'(\psi))(\gamma''(\psi)/\gamma'^3(\psi))']' = 0$ four times with respect to ψ, we obtain

$$c_1\gamma^3(\psi) + c_2\gamma^2(\psi) + c_3\gamma(\psi) + c_4 = \psi, \tag{5.71}$$

where c_1, c_2, c_3, and c_4 are arbitrary constants such that c_1, c_2, and c_3 are not equal to zero simultaneously. Using (5.71) in $D_0(\psi) = 0$, we get

$$c_1 = c_2 = 0, \tag{5.72}$$

and the exact integral for this irrotational flow is given by (5.59) with $\beta(\psi) = 0$ and $\gamma'(\psi) = 1/c_3$.

Summing up, we have the following theorem.

Theorem 5.6. *The streamline pattern $\theta - a_1 r^m - a_2 \ln r$ = constant is permissible for an infinitely conducting MHD aligned, a finitely conducting MHD aligned with $a_1 = 0$ and for non-MHD fluid flows.*

Example 5.7 (Flow with $\theta - r^m$ = constant as streamlines)**.** We let

$$\theta - r^m = \gamma(\psi), \tag{5.73}$$

and we find that

$$\gamma(\psi) = \begin{cases} \dfrac{1}{\mu(m-2)}[\rho - \mu^*\beta_0^2]\psi + \psi_0, & m \neq 2, \\ a\psi + b, & m = 2, \end{cases}$$

$$\beta(\psi) = \begin{cases} \beta_0, & \beta_0 \neq \sqrt{\dfrac{\rho}{\mu^*}},\ m \neq 2, \\ \beta_0, & m = 2, \end{cases}$$

$$v_1 = \begin{cases} \dfrac{\mu(m-2)}{[\rho - \mu^*\beta_0^2]r}, & m \neq 2, \\ \dfrac{1}{ar}, & m = 2, \end{cases}$$

$$v_2 = \begin{cases} \dfrac{\mu m(m-2)}{\rho - \mu^*\beta_0^2} r^{m-1}, & m \neq 2, \\ \dfrac{2r}{a}, & m = 2, \end{cases}$$

$$H_1 = \begin{cases} \dfrac{\mu(m-2)\beta_0}{\rho - \mu^*\beta_0^2}\left[\dfrac{\cos\theta}{r} - mr^{m-1}\sin\theta\right], & m \neq 2, \\ \dfrac{1}{a}\left[\dfrac{\cos\theta}{r} - 2r\sin\theta\right], & m = 2, \end{cases}$$

$$H_2 = \begin{cases} \dfrac{\mu(m-2)\beta_0}{\rho - \mu^*\beta_0^2}\left[\dfrac{\sin\theta}{r} + mr^{m-1}\cos\theta\right], & m \neq 2, \\ \dfrac{1}{a}\left[\dfrac{\sin\theta}{r} + 2r\cos\theta\right], & m = 2, \end{cases}$$

$$p = \begin{cases} \dfrac{\mu^2 m^3(m-2)^2}{2(m-1)(\rho - \mu^*\beta_0^2)} r^{2m-2} - \dfrac{\rho\mu^2(m-2)^2}{2(\rho - \mu^*\beta_0^2)^2}\left[\dfrac{1}{r^2} + m^2 r^{2m-2}\right] + p_0, & m \neq 2,\ m \neq 1, \\ \dfrac{\mu^2}{\rho - \mu^*\beta_0^2}\ln r - \dfrac{\rho\mu^2}{2(\rho - \mu^*\beta_0^2)^2 r^2} + p_0, & m \neq 2,\ m = 1, \\ p_0 - \dfrac{4}{a^2}(\rho - \mu^*\beta_0^2)[\theta - r^2 - b] - \dfrac{\rho}{2a^2}\left[\dfrac{1}{r^2} + 4r^2\right], & m = 2, \end{cases}$$

$$\omega = \begin{cases} \dfrac{\mu m(m-2)}{\rho - \mu^*\beta_0^2} r^{m-2}, & m \neq 2, \\ \dfrac{4}{a}, & m = 2, \end{cases}$$

$$\Omega = \beta_0\omega, \tag{5.74}$$

where ψ_0, $\beta_0 \neq 0$, $a \neq 0$, b, and p_0 are arbitrary constants. The streamline pattern is given in Figure 5.4.

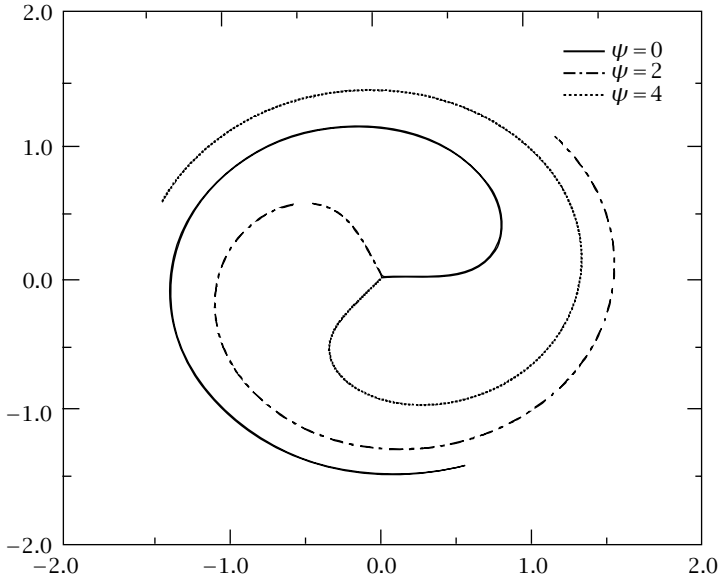

FIGURE 5.4. Streamline pattern for $\theta - r^m = \psi$, $m = 4$.

EXAMPLE 5.8 (Flow with $\theta - a_1 r^2 - a_2 r =$ constant as streamlines)**.** We take

$$\theta - a_1 r^2 - a_2 r = \gamma(\psi), \tag{5.75}$$

and we have

$$\begin{aligned}
\gamma(\psi) &= -\frac{1}{\mu}[\rho - \mu^* \beta_0^2]\psi + \psi_0, \qquad \beta(\psi) = \beta_0, \quad \beta_0 \neq \sqrt{\frac{\rho}{\mu^*}}, \\
v_1 &= -\frac{\mu}{[\rho - \mu^* \beta_0^2] r}, \qquad v_2 = -\frac{\mu}{[\rho - \mu^* \beta_0^2]}(2a_1 r + a_2), \\
H_1 &= -\frac{\mu \beta_0}{[\rho - \mu^* \beta_0^2]}\left[\frac{\cos\theta}{r} - (2a_1 r + a_2)\sin\theta\right], \\
H_2 &= -\frac{\mu \beta_0}{[\rho - \mu^* \beta_0^2]}\left[\frac{\sin\theta}{r} + (2a_1 r + a_2)\cos\theta\right], \\
p &= \frac{\mu^2}{[\rho - \mu^* \beta_0^2]}[6a_1 a_2 r + a_2^2 \ln r - 4a_1(\theta - a_1 r^2 - \psi_0)] \\
&\quad - \frac{\rho \mu^2}{2[\rho - \mu^* \beta_0^2]^2}\left[\frac{1}{r^2} + 4a_1^2 r^2 + 4a_1 a_2 r + a_2^2\right] + p_0, \\
\omega &= -\frac{\mu}{[\rho - \mu^* \beta_0^2]}\left[4a_1 + \frac{a_2}{r}\right], \qquad \Omega = \beta_0 \omega,
\end{aligned} \tag{5.76}$$

where $\psi_0, \beta_0 \neq \sqrt{\rho/\mu^*}$, and p_0 are arbitrary constants. Figure 5.5 shows the flow pattern for this example.

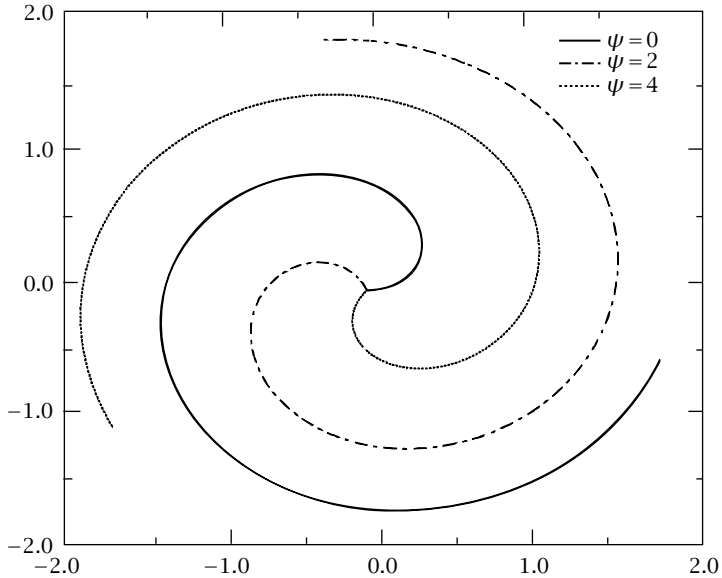

FIGURE 5.5. Streamline pattern for $\theta - a_1 r^2 - a_2 r = \psi$.

EXAMPLE 5.9 (Flow with $\theta - a_1(\ln r)^2 - a_2 \ln r$ = constant as streamlines)**.** We assume that

$$\theta - a_1(\ln r)^2 - a_2 \ln r = \gamma(\psi), \tag{5.77}$$

and we have

$$\begin{aligned}
\gamma(\psi) &= -\frac{1}{2\mu}[\rho - \mu^*\beta_0^2]\psi + \psi_0, \qquad \beta(\psi) = \beta_0,\\
v_1 &= -\frac{2\mu}{[\rho - \mu^*\beta_0^2]r}, \qquad v_2 = -\frac{2\mu}{[\rho - \mu^*\beta_0^2]}\left[\frac{2a_1 \ln r}{r} + \frac{a_2}{r}\right],\\
H_1 &= -\frac{2\mu\beta_0}{[\rho - \mu^*\beta_0^2]r}[\cos\theta - (2a_1 \ln r + a_2)\sin\theta],\\
H_2 &= -\frac{2\mu\beta_0}{[\rho - \mu^*\beta_0^2]r}[\sin\theta + (2a_1 \ln r + a_2)\cos\theta],\\
p &= -\frac{4\mu^2 a_1}{[\rho - \mu^*\beta_0^2]r^2}[2a_1 \ln r + a_1 + a_2] - \frac{2\mu^2\rho}{[\rho - \mu^*\beta^2]^2 r^2}[1 + (2a_1 \ln r + a_2)^2] + p_0,\\
\omega &= -\frac{4\mu a_1}{[\rho - \mu^*\beta_0^2]r^2}, \qquad \Omega = \beta_0\omega,
\end{aligned} \tag{5.78}$$

where p_0 and $\beta_0 \neq \sqrt{\rho/\mu^*}$ are arbitrary constants. The flow pattern is shown in Figure 5.6.

EXAMPLE 5.10 (Flow with $r^2\theta$ = constant as streamlines)**.** We let

$$r^2\theta = \gamma(\psi) \tag{5.79}$$

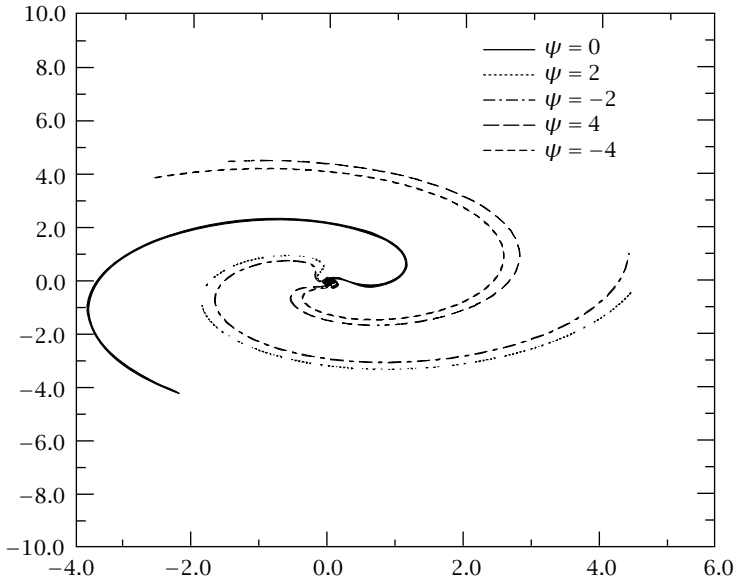

FIGURE 5.6. Streamline pattern for $\theta - a_1(\ln r)^2 - a_2 \ln r = \psi$.

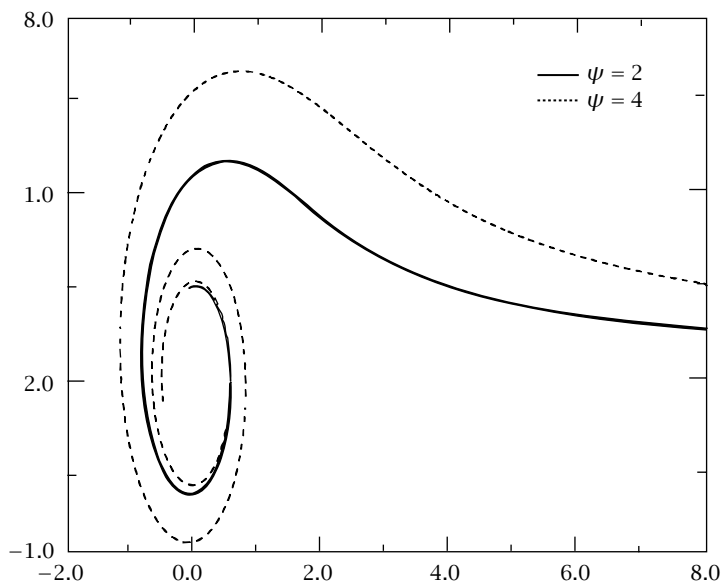

FIGURE 5.7. Streamline pattern for $r^2\theta = \psi$.

and we have

$$\gamma(\psi) = a_1\psi + a_2, \qquad \beta(\psi) = \sqrt{\frac{\rho}{\mu^*}},$$

$$
\begin{aligned}
v_1 &= \frac{r}{a_1}, \qquad v_2 = -\frac{2r\theta}{a_1}, \\
H_1 &= \sqrt{\frac{\rho}{\mu^*}}\frac{r}{a_1}[\cos\theta + 2\theta\sin\theta], \qquad H_2 = \sqrt{\frac{\rho}{\mu^*}}\frac{r}{a_1}[\sin\theta - 2\theta\cos\theta], \\
p &= \frac{4\mu}{a_1}\ln r - \frac{\rho r^2}{2a_1^2}(1+4\theta^2) + p_0, \\
\omega &= -\frac{4\theta}{a_1}, \qquad \Omega = \sqrt{\frac{\rho}{\mu^*}}\omega,
\end{aligned}
\tag{5.80}
$$

where $a_1 \neq 0$, a_2, and p_0 are arbitrary constants. The streamlines are shown in Figure 5.7.

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