ON THE FEKETE-SZEGÖ PROBLEM

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ABSTRACT. Let $f(z) = z + a_2 z^2 + a_3 z^3 + \dots$ be an analytic function in the open unit disk. A sharp upper bound is obtained for $|a_3 - \mu a_2^2|$ by using the classes of strongly starlike functions of order β and type α when $\mu \geq 1$.

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1. Introduction. Let $\mathcal{A}$ denote the family of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1.1)$$

which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. Further, let $\mathcal{S}$ denote the class of functions which are univalent in $\mathcal{U}$. A function $f(z)$ belonging to $\mathcal{A}$ is said to be strongly starlike of order β and type α in $\mathcal{U}$, and denoted by $\tilde{\mathcal{S}}_{\alpha}^*(\beta)$ if it satisfies

$$\left| \arg \left(\frac{zf'(z)}{f(z)} - \alpha \right) \right| < \frac{\pi}{2} \beta \quad (z \in \mathcal{U}) \quad (1.2)$$

for some α ($0 \leq \alpha < 1$) and β ($0 < \beta \leq 1$). If $f(z) \in \mathcal{A}$ satisfies

$$\left| \arg \left(1 + \frac{zf''(z)}{f'(z)} - \alpha \right) \right| < \frac{\pi}{2} \beta \quad (z \in \mathcal{U}) \quad (1.3)$$

for some α ($0 \leq \alpha < 1$) and β ($0 < \beta \leq 1$), then we say that $f(z)$ is strongly convex of order β and type α in $\mathcal{U}$, and we denote by $\tilde{\mathcal{C}}_{\alpha}(\beta)$ the class of all such functions (see also Srivastava and Owa [16]). For the class $\mathcal{S}$ of analytic univalent functions, Fekete-Szegö [6] obtained the maximum value of $|a_3 - \mu a_2^2|$ when μ is real. For various functions of $\mathcal{S}$, the upper bound for $|a_3 - \mu a_2^2|$ is investigated by many different authors including [1, 2, 3, 4, 5, 7, 10, 11, 12, 13, 14, 17].

In this paper, we obtain sharp upper bounds for $|a_3 - \mu a_2^2|$ when f belonging to the classes of functions defined as follows.

DEFINITION 1.1. Let $0 \leq \alpha < 1$, $\beta > 0$ and let $f \in \mathcal{A}$. Then $f \in \mathcal{M}(\alpha, \beta)$ if and only if there exist $g \in \tilde{\mathcal{S}}_{\alpha}^*(\beta)$ such that

$$\operatorname{Re} \left(\frac{zf'(z)}{g(z)} \right) > 0 \quad (z \in \mathcal{U}), \quad (1.4)$$

and $f \in \mathcal{G}(\alpha, \beta)$ if and only if there exists $g \in \tilde{\mathcal{C}}_{\alpha}(\beta)$ and satisfy condition (1.4) with $g(z) = z + b_2 z^2 + b_3 z^3 + \dots$.

Note that $\mathcal{M}(0, \beta) = \mathcal{H}(\beta)$ is the class of close-to-convex functions defined in [3] and $\mathcal{M}(0, 1) = \mathcal{H}(1)$ is the class of normalized close-to-convex functions defined by Kaplan [9].

2. Main results. In order to derive our main results, we have to recall here the following lemma [15].

LEMMA 2.1. *Let $h \in \mathcal{P}$, that is, h be analytic in $\mathcal{U}$ and be given by $h(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \dots$, and $\operatorname{Re} h(z) > 0$ for $z \in \mathcal{U}$, then*

$$\left| c_2 - \frac{c_1^2}{2} \right| \leq 2 - \frac{|c_1^2|}{2}. \quad (2.1)$$

THEOREM 2.2. *Let $f(z) \in \mathcal{M}(\alpha, \beta)$ and be given by (1.1). Then for $0 \leq \alpha < 1$, $\beta \geq 1$, and $\mu \geq 1$ we have the sharp inequality*

$$|a_3 - \mu a_2^2| \leq \frac{2\beta^2(\mu-1) + \alpha\beta^2(8-2\alpha-3\mu)}{(1-\alpha)^2(2-\alpha)} + \frac{(2\beta+1-\alpha)(3\mu-2)}{3(1-\alpha)}. \quad (2.2)$$

PROOF. Let $f(z) \in \mathcal{M}(\alpha, \beta)$. It follows from (1.4) that

$$zf'(z) = g(z)q(z), \quad (2.3)$$

for $z \in \mathcal{U}$, with $q \in \mathcal{P}$ given by $q(z) = 1 + q_1 z + q_2 z^2 + q_3 z^3 + \dots$. Equating coefficients, we obtain

$$2a_2 = q_1 + b_2, \quad 3a_3 = q_2 + b_2 q_1 + b_3. \quad (2.4)$$

Also, it follows from (1.2) that

$$zg'(z) - \alpha g(z) = g(z)(p(z))^\beta, \quad (2.5)$$

where $z \in \mathcal{U}$, $p \in \mathcal{P}$, and $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \dots$. Thus equating coefficients, we obtain

$$(1-\alpha)b_2 = \beta p_1, \quad (2-\alpha)b_3 = \beta \left(p_2 + \frac{\beta(3-\alpha) + \alpha - 1}{2(1-\alpha)} p_1^2 \right). \quad (2.6)$$

From (2.4) and (2.6), we have

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{1}{3} \left(q_2 - \frac{1}{2} q_1^2 \right) + \frac{2-3\mu}{12} q_1^2 + \frac{\beta}{3(2-\alpha)} \left(p_2 - \frac{1}{2} p_1^2 \right) \\ &\quad + \frac{\beta^2[6(1-\mu) + \alpha(2\alpha+3\mu-8)]}{12(1-\alpha)^2(2-\alpha)} p_1^2 + \frac{\beta(2-3\mu)}{6(1-\alpha)} p_1 q_1. \end{aligned} \quad (2.7)$$

Assume that $a_3 - \mu a_2^2$ is positive. Thus we now estimate $\operatorname{Re}(a_3 - \mu a_2^2)$, so from (2.7) and by using Lemma 2.1 and letting $p_1 = 2re^{i\theta}$, $q_1 = 2Re^{i\phi}$, $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$, and $0 \leq \phi < 2\pi$, we obtain

$$\begin{aligned}
3 \operatorname{Re}(a_3 - \mu a_2^2) &= \operatorname{Re}\left(q_2 - \frac{1}{2}q_1^2\right) + \frac{2-3\mu}{4} \operatorname{Re} q_1^2 + \frac{\beta}{(2-\alpha)} \operatorname{Re}\left(p_2 - \frac{1}{2}p_1^2\right) \\
&\quad + \frac{\beta^2[(6+2\alpha^2+3\alpha\mu)-(6\mu+8\alpha)]}{4(1-\alpha)^2(2-\alpha)} \operatorname{Re} p_1^2 + \frac{\beta(2-3\mu)}{2(1-\alpha)} \operatorname{Re} p_1 q_1. \\
&\leq 2(1-R^2) + (2-3\mu)R^2 \cos 2\phi + \frac{2\beta}{2-\alpha}(1-r^2) \\
&\quad + \frac{\beta^2[(6+2\alpha^2+3\alpha\mu)-(6\mu+8\alpha)]}{(1-\alpha)^2(2-\alpha)} r^2 \cos 2\theta + \frac{2\beta(2-3\mu)}{1-\alpha} rR \cos(\theta+\phi) \\
&\leq (3\mu-4)R^2 + \frac{2\beta(3\mu-2)}{1-\alpha} rR \\
&\quad + \frac{6\beta^2(\mu-1) + \alpha\beta^2(8-2\alpha-3\mu) - 2\beta(1-\alpha)^2}{(1-\alpha)^2(2-\alpha)} r^2 + \frac{2(\beta-\alpha)+4}{2-\alpha} \\
&= \Psi(r, R).
\end{aligned} \tag{2.8}$$

Letting α , β , and μ fixed and differentiating $\Psi(r, R)$ partially when $0 \leq \alpha < 1$, $\beta \geq 1$, and $\mu \geq 1$, we observe that

$$\begin{aligned}
\Psi_{rr}\Psi_{RR} - (\Psi_{rR})^2 &= 4\beta[4\beta + 2 + \alpha(2\alpha\beta + 2\alpha - 4 - 7\beta)] \\
&\quad - 3\beta\mu[6\beta + 2 + \alpha(2\alpha\beta + 2\alpha - 4 - 8\beta)] < 0.
\end{aligned} \tag{2.9}$$

Therefore, the maximum of $\Psi(r, R)$ occurs on the boundaries. Thus the desired inequality follows by observing that

$$\Psi(r, R) \leq \Psi(1, 1) = \frac{6\beta^2(\mu-1) + \alpha\beta^2(8-2\alpha-3\mu)}{(1-\alpha)^2(2-\alpha)} + \frac{(2\beta+1-\alpha)(3\mu-2)}{1-\alpha}. \tag{2.10}$$

The equality for (2.2) is attained when $p_1 = q_1 = 2i$ and $q_1 = q_2 = -2$.

Letting $\alpha = 0$ in Theorem 2.2, we have the result given by Jahangiri [8]. $\square$

COROLLARY 2.3. *Let $f(z) \in \mathcal{H}(\beta)$ and be given by (1.1). Then for $\beta \geq 1$, and $\mu \geq 1$, we have the sharp inequality*

$$|a_3 - \mu a_2^2| \leq \beta^2(\mu-1) + \frac{(2\beta+1)(3\mu-2)}{3}. \tag{2.11}$$

THEOREM 2.4. *Let $f(z) \in \mathcal{G}(\alpha, \beta)$ and be given by (1.1). Then for $0 \leq \alpha < 1$, $\beta \geq 1$, and $\mu \geq 1$, we have the sharp inequality*

$$|a_3 - \mu a_2^2| \leq \frac{6\beta^2(3\mu-4) + \alpha\beta^2(32-8\alpha-9\mu)}{36(1-\alpha)^2(2-\alpha)} + \frac{(\beta+1-\alpha)(3\mu-2)}{3(1-\alpha)}. \tag{2.12}$$

PROOF. Let $f(z) \in \mathcal{G}(\alpha, \beta)$. It follows from (1.3) that

$$zg''(z) + (1-\alpha)g'(z) = g'(z)(p(z))^\beta, \tag{2.13}$$

where $z \in \mathcal{U}$, $p \in \mathcal{P}$, and $p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \dots$. Thus equating coefficients, we obtain

$$2(1-\alpha)b_2 = \beta p_1, \quad 3(2-\alpha)b_3 = \beta\left(p_2 + \frac{\beta(3-\alpha) + \alpha - 1}{2(1-\alpha)}p_1^2\right). \tag{2.14}$$

From (2.4) and (2.14) and proceeding as in the proof of Theorem 2.2, we get

$$\begin{aligned} 3\operatorname{Re}(a_3 - \mu a_2^2) &\leq (3\mu - 4)R^2 + \frac{\beta(3\mu - 2)}{1 - \alpha}rR + \frac{2(\beta - 3\alpha) + 12}{3(2 - \alpha)} \\ &\quad + \frac{6\beta^2(3\mu - 4) + \alpha\beta^2(32 - 8\alpha - 9\mu) - 8\beta(1 - \alpha)^2}{12(1 - \alpha)^2(2 - \alpha)}r^2 \\ &= \Phi(r, R). \end{aligned} \quad (2.15)$$

Letting α , β and μ fixed and differentiating $\Phi(r, R)$ partially when $0 \leq \alpha < 1$, $\beta \geq 1$, and $\mu \geq 1$, we have

$$\begin{aligned} \Phi_{rr}\Phi_{RR} - (\Phi_{rR})^2 &= 4\beta[18\beta + 8 + \alpha(8\alpha\beta + 8\alpha - 16 - 29\beta)] \\ &\quad - 3\beta\mu[24\beta + 8 + \alpha(8\alpha\beta + 8\alpha - 16 - 32\beta)] < 0. \end{aligned} \quad (2.16)$$

Therefore, the maximum of $\Phi(r, R)$ occurs on the boundaries. Thus the desired inequality (2.12) follows by observing that

$$\Phi(r, R) \leq \Phi(1, 1) = \frac{6\beta^2(3\mu - 4) + \alpha\beta^2(32 - 8\alpha - 9\mu)}{12(1 - \alpha)^2(2 - \alpha)} + \frac{(\beta + 1 - \alpha)(3\mu - 2)}{1 - \alpha}. \quad (2.17)$$

The equality in (2.12) is attained on choosing $p_1 = q_1 = 2i$ and $q_1 = q_2 = -2$. This completes the proof of Theorem 2.4. $\square$

COROLLARY 2.5. *Let $f(z) \in \mathcal{G}(0, \beta)$ and be given by (1.1). Then for $\beta \geq 1$, and $\mu \geq 1$, we have the sharp inequality*

$$|a_3 - \mu a_2^2| \leq \frac{1}{12} [(3\mu - 2)(\beta + 2)^2 - 2\beta^2]. \quad (2.18)$$

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