ON S-CLUSTER SETS AND S-CLOSED SPACES

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ABSTRACT. A new type of cluster sets, called S-cluster sets, of functions and multifunctions between topological spaces is introduced, thereby providing a new technique for studying S-closed spaces. The deliberation includes an explicit expression of S-cluster set of a function. As an application, characterizations of Hausdorff and S-closed topological spaces are achieved via such cluster sets.

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1. Introduction. The theory of cluster sets was developed long ago, and was initially aimed at the investigations of real and complex function theory (see [15]). A comprehensive collection of works in this direction can be found in the classical book of Collingwood and Lohwater [1]. Weston [14] was the first to initiate the corresponding theory for functions between topological spaces basically for studying compactness. Many others (e.g., see [3, 4, 6]) followed suit with cluster sets, \( \theta \)-cluster sets and \( \delta \)-cluster sets of functions and multifunctions, ultimately implicating different covering properties, among other things.

The present paper is intended for the introduction of a new type of cluster sets, called S-cluster sets, which provides a new technique for the study of S-closedness of topological spaces. It is shown that such cluster sets of suitable function can characterize Hausdorffness. Finally, we achieve, as our prime motivation, certain characterizations of an S-closed space.

In what follows, \( X \) and \( Y \) denote topological spaces, and \( f : X \to Y \) is a function from \( X \) into \( Y \). By a multifunction \( F : X \to Y \) we mean, as usual, a function mapping points of \( X \) into the nonempty subsets of \( Y \). The set of all open sets of \((X, \tau)\), each containing a given point \( x \) of \( X \), is denoted by \( \tau(x) \). A set \( A (\subseteq X) \) is called semi-open [5] if for some open set \( U, U \subseteq A \subseteq \text{cl} U \), where \( \text{cl} U \) denotes the closure of \( U \) in \( X \). The set of all semi-open sets of \( X \), each containing a given subset \( A \) of \( X \), is denoted by \( \text{SO}(A) \), in particular, if \( A = \{x\} \), we write \( \text{SO}(x) \) instead of \( \text{SO}(\{x\}) \). The complements of semi-open sets are called semi-closed. For any subset \( A \) of \( X \), the \( \theta \)-closure [13] (\( \theta \)-semiclosure [8]) of \( A \), denoted by \( \theta \text{-cl} A \) (respectively, \( \theta_s \text{-cl} A \)), is the set of all points \( x \) of \( X \) such that for every \( U \in \tau(x) \) (respectively, \( U \in \text{SO}(x) \)), \( \text{cl} U \cap A \neq \emptyset \). The set \( A \) is called \( \theta \)-closed [13] (\( \theta \)-semiclosed [8]) if \( A = \theta \text{-cl} A \) (respectively, \( A = \theta_s \text{-cl} A \)). It is known [9] that \( \theta \text{-cl} A \) need not be \( \theta \)-closed, but it is so if \( A \) is open. A nonvoid collection \( \Omega \) of nonempty subsets of a space \( X \) is called a grill [12] if

(i) \( A \in \Omega \) and \( B \supseteq A \Rightarrow B \in \Omega \),
A filterbase $\mathcal{F}$ on a space $X$ is said to $\theta_S$-adhere [11] at a point $x$ of $X$, denoted as $x \in \theta_S$-ad $\mathcal{F}$, if $x \in \cap \{\theta_S F : F \in \mathcal{F}\}$. A grill $\Omega$ on $X$ is said to $\theta_S$-converge to a point $x$ of $X$, if to each $U \in SO(x)$, there corresponds some $G \in \Omega$ with $G \subseteq cl U$. A set $A$ in a space $X$ is said to be $S$-closed relative to $X$ [7] if for every cover $\mathcal{U}$ of $A$ by semi-open sets of $X$, there exists a finite subfamily $\mathcal{U}_0$ of $\mathcal{U}$ such that $A \subseteq \cup \{cl U : U \in \mathcal{U}_0\}$. If, in addition, $A = X$, then $X$ is called an $S$-closed space [11].

2. Main theorem and associated results. We begin by introducing $S$-cluster set of a function and of a multifunction between two topological spaces.

**Definition 2.1.** Let $F : X \to Y$ be a multifunction and $x \in X$. Then the $S$-cluster set of $F$ at $x$, denoted by $S(F, x)$, is defined to be the set $\cap \{\theta_S F : U \in SO(x)\}$. Similarly, for any function $f : X \to Y$, the $S$-cluster set $S(f, x)$ of $f$ at $x$ is given by $\cap \{\theta_S f : U \in SO(x)\}$.

In the next theorem, we characterize the $S$-cluster sets of functions between topological spaces.

**Theorem 2.2.** For any function $f : X \to Y$, the following statements are equivalent.

(a) $y \in S(f, x)$.
(b) The filterbase $f^{-1}(cl \tau(y))$ $\theta_S$-adheres at $x$.
(c) There is a grill $\Omega$ on $X$ such that $\Omega \theta_S$-converges to $x$ and $y \in \cap \{\theta_S f : G \in \Omega\}$.

**Proof.** (a)$\Rightarrow$(b). $y \in S(f, x) \Rightarrow$ for each $W \in SO(x)$ and each $V \in \tau(y)$, $cl V \cap f(cl W) \neq \emptyset$ $\Rightarrow$ for each $W \in SO(x)$ and each $V \in \tau(y)$, $f^{-1}(cl V) \cap cl W \neq \emptyset$. This ensures that the collection $\{f^{-1}(cl V) : V \in \tau(y)\}$ (which can easily be seen to be a filterbase on $X$) $\theta_S$-adheres at $x$.

(b)$\Rightarrow$(c). Let $\mathcal{F}$ be the filter generated by the filterbase $f^{-1}(cl \tau(y))$. Then $\Omega = \{G \subseteq X : G \cap F \neq \emptyset, \text{ for each } F \in \mathcal{F}\}$ is a grill on $X$. By the hypothesis, for each $U \in SO(x)$ and each $V \in \tau(y)$, $cl U \cap f^{-1}(cl V) \neq \emptyset$. Hence, $F \cap cl U \neq \emptyset$ for each $F \in \mathcal{F}$ and each $U \in SO(x)$. Consequently, $cl U \in \Omega$ for all $U \in SO(x)$, which proves that $\Omega \theta_S$-converges to $x$. Now, the definition of $\Omega$ yields that $f(G) \cap cl W \neq \emptyset$ for all $W \in \tau(y)$ and all $G \in \Omega$, i.e., $y \in \theta_S f(G)$ for all $G \in \Omega$. Hence, $y \in \cap \{\theta_S f(G) : G \in \Omega\}$.

(c)$\Rightarrow$(a). Let $\Omega$ be a grill on $X$ such that $\Omega \theta_S$-converges to $x$, and $y \in \cap \{\theta_S f(G) : G \in \Omega\}$. Then $\{cl U : U \in SO(x)\} \subseteq \Omega$ and $y \in \theta_S f(cl U)$ for each $G \in \Omega$. Hence, in particular, $y \in \theta_S f(cl U)$ for all $U \in SO(x)$. So, $y \in \cap \{\theta_S f(cl U) : U \in SO(x)\} = S(f, x)$.

In what follows, we show that $S$-cluster sets of a function may be used to ascertain the Hausdorffness of the codomain space.

**Theorem 2.3.** Let $f : X \to Y$ be a function on a topological space $X$ onto a topological space $Y$. Then $Y$ is Hausdorff if $S(f, x)$ is degenerate for each $x \in X$.

**Proof.** Let $y_1, y_2 \in Y$ such that $y_1 \neq y_2$. As $f$ is a surjection, there are $x_1, x_2 \in X$ such that $f(x_i) = y_i$ for $i = 1, 2$. Now, since $S(f, x)$ is degenerate for each $x \in X$, $y_2 = f(x_2) \notin S(f, x_1)$. Thus, there are $V \in \tau(y_2)$ and $U \in SO(x_1)$ such that $cl V \cap f(cl U) = \emptyset$. Therefore, $y_1 \neq y_2$. Hence, $Y$ is Hausdorff.
\( \emptyset \), i.e., \( f(\text{cl } U) \subseteq Y - \text{cl } V \). Then the open sets \( Y - \text{cl } V \) and \( V \) strongly separate \( y_1 \) and \( y_2 \) in \( Y \), which proves that \( Y \) is Hausdorff.

**Remark 2.4.** We note that the converse of the above theorem is false. For example, consider the identity map \( f : (\mathbb{R}, \tau) \to (\mathbb{R}, \sigma) \), where \( \tau \) and \( \sigma \), respectively, denote the cofinite topology and the usual topology on the set \( \mathbb{R} \) of real numbers. Then \( S(f, x) = \mathbb{R} \) for each \( x \in \mathbb{R} \), though \((\mathbb{R}, \sigma)\) is a \( T_3 \)-space.

In order to obtain the converse, we introduce the following class of functions.

**Definition 2.5.** A function \( f : X \to Y \) is called \( \theta \)-irresolute on \( X \) if for each \( x \in X \) and each semi-open set \( V \) containing \( f(x) \), there is a semi-open set \( U \) containing \( x \) such that \( f(\text{cl } U) \subseteq V \).

**Theorem 2.6.** Let \( f : X \to Y \) be a \( \theta \)-irresolute function with \( Y \) a Hausdorff space. Then \( S(f, x) \) is degenerate for each \( x \in X \).

**Proof.** Let \( x \in X \). As \( f \) is \( \theta \)-irresolute on \( X \), for any \( V \in \text{SO}(f(x)) \), there is \( U \in \text{SO}(x) \) such that \( f(\text{cl } U) \subseteq V \). Then \( S(f, x) = \cap \{ \theta\text{-cl } f(\text{cl } U) : U \in \text{SO}(x) \} \subseteq \cap \{ \theta\text{-cl } V : V \in \text{SO}(f(x)) \} \). Let \( y \in Y \) with \( y \neq f(x) \). As \( Y \) is Hausdorff, there are disjoint open sets \( U, W \) with \( y \in U, f(x) \in W \). Obviously, as \( U \cap \text{cl } W = \emptyset \), \( y \notin \text{cl } W = \theta\text{-cl } W \). As \( W \in \tau(f(x)) \subseteq \text{SO}(f(x)) \), \( y \notin \cap \{ \theta\text{-cl } V : V \in \text{SO}(f(x)) \} \) and hence \( y \notin S(f, x) \). Thus, \( S(f, x) = \{ f(x) \} \).

Combining the last two results, we get the following characterization for the Hausdorffness of the codomain space of a kind of function in terms of the degeneracy of its \( S \)-cluster set.

**Corollary 2.7.** Let \( f : X \to Y \) be a \( \theta \)-irresolute function on \( X \) onto \( Y \). Then the space \( Y \) is Hausdorff if and only if \( S(f, x) \) is degenerate for each \( x \) of \( X \).

We have just seen that degeneracy of the \( S \)-cluster set of an arbitrary function is a sufficient condition for the Hausdorffness of the codomain space. We thus like to examine some other situations when the \( S \)-cluster sets are degenerate, thereby ensuring the Hausdorffness of the codomain space of the function concerned. To this end, we recall that a topological space \((X, \tau)\) is almost regular [10] if for every regular closed set \( A \) in \( X \) and for each \( x \notin A \), there exist disjoint open sets \( U \) and \( V \) such that \( x \in U \) and \( A \subseteq V \). It is known that in an almost regular space \( X \), \( \theta\text{-cl } A \) is \( \theta\)-closed for each \( A \subseteq X \). A function \( f : X \to Y \) carrying \( \theta \)-closed sets of \( X \) into \( \theta \)-closed sets of \( Y \) is called a \( \theta \)-closed function [2].

**Theorem 2.8.** Let \( f : X \to Y \) be a \( \theta \)-closed map from an almost regular space into a space \( Y \). If \( f^{-1}(y) \) is \( \theta \)-closed in \( X \) for all \( y \in Y \), then \( S(f, x) \) is degenerate for each \( x \in X \).

**Proof.** We have \( S(f, x) = \cap \{ \theta\text{-cl } f(\text{cl } U) : U \in \text{SO}(x) \} \subseteq \cap \{ \theta\text{-cl } f(\theta\text{-cl } U) : U \in \text{SO}(x) \} \). As \( X \) is almost regular, \( \theta\text{-cl } U \) is \( \theta \)-closed for all \( U \in \text{SO}(x) \). Now, since \( f \) is a \( \theta \)-closed map, \( \theta\text{-cl } f(\theta\text{-cl } U) = f(\theta\text{-cl } U) \) for each \( U \in \text{SO}(x) \). Thus, \( S(f, x) \subseteq \cap \{ f(\theta\text{-cl } U) : U \in \text{SO}(x) \} \). Now, let \( y \in Y \) such that \( y \neq f(x) \). Then since \( f^{-1}(y) \) is \( \theta \)-closed and \( x \notin f^{-1}(y) \), there is some \( P \in \tau(x) \) such that \( \text{cl } P \cap f^{-1}(y) = \emptyset \). So,
For that in view of what we have deduced above, we conclude that \( y \notin S(f, x) \), which proves that \( S(f, x) \) is degenerate.

**Theorem 2.9.** Let \( f : X \to X \) be a \( \theta \)-closed injection on an almost regular Hausdorff space \( X \) into \( Y \). Then \( S(f, x) \) is degenerate for each \( x \in X \).

**Proof.** As \( X \) is almost regular and \( f \) is a \( \theta \)-closed map, we have \( \theta \text{-cl } f(\theta \text{-cl } U) = f(\theta \text{-cl } U) \) for any \( U \in SO(x) \) and, hence,

\[
S(f, x) = \cap \{ \theta \text{-cl } f(\text{cl } U) : U \in SO(x) \} \subseteq \cap \{ \theta \text{-cl } f(\theta \text{-cl } U) : U \in SO(x) \} = \cap \{ f(\theta \text{-cl } U) : U \in SO(x) \}.
\]  

(2.1)

For \( x, x_1 \in X \) with \( x \neq x_1 \), \( f(x) \neq f(x_1) \) as \( f \) is injective. By the Hausdorffness of \( X \), there are disjoint open sets \( U, V \) in \( X \) with \( x \in U, x_1 \in V \). Obviously, \( U \cap V = \emptyset \).

\( SO, x_1 \notin \theta \text{-cl } U \) and hence \( f(x_1) \notin f(\theta \text{-cl } U) \). Since \( U \in \tau(x) \subseteq SO(x) \), equation (2.1) yields \( f(x_1) \notin S(f, x) \). Thus, \( S(f, x) \) is degenerate for each \( x \in X \).

The above theorem is equivalent to the following apparently weaker result when \( X \) is regular.

**Theorem 2.10.** If \( f : X \to Y \) is a \( \theta \)-closed injection on a \( T_3 \) space \( X \) into a space \( Y \), then \( S(f, x) \) is degenerate for each \( x \in X \).

**Proof.** It is known that in a regular space \( X, \theta \text{-cl } U = \text{cl } U \) for any \( U \subseteq X \). Since \( X \) is \( T_3 \) and \( f \) is a \( \theta \)-closed injection, \( \{ f(x) \} \subseteq S(f, x) = \cap \{ f(\text{cl } U) : U \in SO(x) \} \subseteq \cap \{ f(\text{cl } U) : U \in \tau(x) \} = \{ f(x) \} \).

Note that the above result is indeed equivalent to that of Theorem 2.9 follows from the following considerations: for any subset \( A \) of a topological space \( (X, \tau) \), \( \theta \)-closure of \( A \) in \( (X, \tau) \) is the same as that in \( (X, \tau_0) \), where \( (X, \tau_0) \) denotes the semiregularization space [9] of \( (X, \tau) \). Moreover, it is known [9] that \( (X, \tau) \) is Hausdorff (almost regular) if and only if \( (X, \tau_0) \) is Hausdorff (respectively, regular). Now, since \( SO(X, \tau) \subseteq SO(X, \tau_0) \), it follows that \( S(f, x) = S(f : (X, \tau) \to Y, x) \subseteq S(f : (X, \tau_0) \to Y, x) \). So, \( S(f, x) \) is degenerate for each \( x \in X \) if \( (X, \tau) \) is an almost regular Hausdorff space and \( f : X \to Y \) is a \( \theta \)-closed injection.

A sort of degeneracy condition for the \( S \)-cluster set of a multifunction with \( \theta \)-closed graph is now obtained.

**Theorem 2.11.** For a multifunction \( F : X \to Y \), if \( F \) has a \( \theta \)-closed graph, then \( S(F, x) = F(x) \).

**Proof.** For any \( y \in S(F, x) \), \( \text{cl } W \cap F(\text{cl } U) \neq \emptyset \) and hence \( F^{-1}(\text{cl } W) \cap \text{cl } U \neq \emptyset \) for each \( U \in SO(x) \) and each \( W \in \tau(y) \), where, as usual, \( F^{-1}(B) = \{ x \in X : F(x) \cap B \neq \emptyset \} \) for any subset \( B \) of \( Y \). Then for any basic open set \( M \times N \) in \( X \times Y \) containing \( (x, y) \), \( F^{-1}(\text{cl } N) \cap \text{cl } M \neq \emptyset \). So, \( (\text{cl } M \times \text{cl } N) \cap \text{cl } G(F) \neq \emptyset \) and hence \( \text{cl } (M \times N) \cap \text{cl } G(F) \neq \emptyset \), where \( G(F) = \{(x, y) \in X \times Y : y \in F(x)\} \) denotes the graph of \( F \). So, \( (x, y) \in \theta \text{-cl } G(F) = G(F) \) (as \( G(F) \) is \( \theta \)-closed). Hence, \( (x, y) \in [G(F) \cap (\{x\} \times Y)] \) so that \( y \in p_2[\{\{x\} \times Y) \cap G(F)] = F(x) \), where \( p_2 : X \times Y \to Y \) is the second projection map.
It is obvious that $F(x) \subseteq S(F,x)$ for each $x \in X$. Hence, $S(F,x) = F(x)$ holds for all $x \in X$.

The next result serves as a partial converse of the above one.

**Theorem 2.12.** For a multifunction $F : X \to Y$, if $S(F,x) = F(x)$ for each $x \in X$, then the graph $G(F)$ of $F$ is $\theta$-semiclosed (and hence semi-closed).

**Proof.** Let $(x,y) \in X \times Y - G(F)$. Now, $y \not\in F(x) = S(F,x) \Rightarrow$ there exist some $W \in SO(x)$ and some $V \in \tau(y)$ such that $\text{cl} V \cap F(\text{cl} W) = \emptyset \Rightarrow (\text{cl} W \times \text{cl} V) \cap G(F) = \emptyset \Rightarrow \text{cl}(W \times V) \cap G(F) = \emptyset$. As $W \times V$ is a semi-open set in $X \times Y$ containing $(x,y)$, $(x,y) \not\in \theta_{S^{-}\text{cl}} G(F)$. Hence, $G(F)$ is $\theta$-semi-closed.

We now turn our attention to the characterizations of $S$-closedness via $S$-cluster sets. We need the following lemmas for this purpose.

**Lemma 2.13.** A set $A$ in a topological space $X$ is an $S$-closed set relative to $X$ if and only if for every filterbase $\mathcal{F}$ on $X$ with $F \cap C \neq \emptyset$ for all $F \in \mathcal{F}$ and for all $C \in SO(A)$, $A \cap \theta_S^{-}\text{ad} \mathcal{F} \neq \emptyset$.

**Proof.** Let $A$ be an $S$-closed set relative to $X$ and let $\mathcal{F}$ be a filterbase on $X$ with the stated property. If possible, suppose that $A \cap \theta_S^{-}\text{ad} \mathcal{F} = \emptyset$. Then for each $x \in A$, there is a semi-open set $V(x)$ in $X$ containing $x$ such that $\text{cl}(V(x)) \cap F(x) = \emptyset$ for some $F(x) \in \mathcal{F}$. Now, $\{V(x) : x \in A\}$ is a cover of $A$ by semi-open sets of $X$. By the $S$-closedness of $A$ relative to $X$, there is a finite subset $A^*$ of $A$ such that $A \subseteq \bigcup \{\text{cl}(V(x)) : x \in A^*\}$. Choose $F^* \in \mathcal{F}$ such that $F^* \subseteq \bigcap \{F(x) : x \in A^*\}$. Then $F^* \cap (\bigcup \{\text{cl}(V(x)) : x \in A^*\}) = \emptyset$, i.e., $F^* \cap (\bigcup \{V(x) : x \in A^*\}) = \emptyset$. Now, as $\bigcup \{V(x) : x \in A^*\}$ is a semi-open set in $X$, $\bigcup \{\text{cl}(V(x)) : x \in A^*\} \in SO(A)$, a contradiction.

Conversely, assume that $A$ is not $S$-closed relative to $X$. Then for some cover $\{U_\alpha : \alpha \in \Lambda\}$ of $A$ by semi-open sets of $X$, $A \not\subseteq \bigcup_{\alpha \in \Lambda_0} \text{cl} U_\alpha$ for each finite subset $\Lambda_0$ of $\Lambda$. So, $\mathcal{F} = \{A - \bigcup_{\alpha \in \Lambda_0} \text{cl} U_\alpha : \Lambda_0 \text{ is a finite subset of } \Lambda\}$ is filterbase on $X$, with $F \cap C \neq \emptyset$, for each $F \in \mathcal{F}$ and each $C \in SO(A)$. But $A \cap \theta_S^{-}\text{ad} \mathcal{F} = \emptyset$.

**Lemma 2.14** [8, 11]. (a) A topological space $X$ is $S$-closed if and only if every filterbase $\theta_S^{-}\text{ad}$-adheres in $X$.

(b) Any $\theta$-semiclosed subset of an $S$-closed space $X$ is $S$-closed relative to $X$.

**Definition 2.15.** For a function or a multifunction $F : X \to Y$ and a set $A \subseteq X$, the notation $S(F,A)$ stands for the set $\bigcup \{S(F,x) : x \in A\}$.

**Theorem 2.16.** For any topological space $X$, the following statements are equivalent.

(a) $X$ is $S$-closed.

(b) $S(F,A) \supseteq \cap \{\theta^{-}\text{cl} F(U) : U \in SO(A)\}$ for each $\theta$-semiclosed subset $A$ of $X$, for each topological space $Y$ and each multifunction $F : X \to Y$.

(c) $S(F,A) \supseteq \cap \{\theta^{-}\text{cl} F(U) : U \in SO(A)\}$ for each $\theta$-semiclosed subset $A$ of $X$, for each topological space $Y$ and each multifunction $F : X \to Y$.

**Proof.** (a)⇒(b). Let $A$ be any $\theta$-semiclosed subset of $X$, where $X$ is $S$-closed. Then by Lemma 2.14(b), $A$ is $S$-closed relative to $X$. Now, let $z \in \cap \{\theta^{-}\text{cl} F(W) : W \in SO(A)\}$. 

Then for all $W \in \tau(z)$ and for each $U \in \mathrm{SO}(A)$, $\cl W \cap F(U) \neq \emptyset$, i.e., $F^{-}(\cl W) \cap U \neq \emptyset$.

Thus, $\mathcal{F} = \{F^{-}(\cl W) : W \in \tau(z)\}$ is clearly a filterbase on $X$, satisfying the condition of Lemma 2.13. Hence, $x \in A \cap \theta_{3}$-ad $\mathcal{F}$. Then $x \in A$, and for all $U \in \mathrm{SO}(x)$ and each $W \in \tau(z)$, $\cl U \cap F^{-}(\cl W) \neq \emptyset$, i.e., $F(\cl U) \cap \cl W \neq \emptyset \Rightarrow z \in S(F,x) \subseteq S(F,A)$.

(b) $\Rightarrow$ (a). Obvious.

(c) $\Rightarrow$ (a). In order to show that $X$ is $S$-closed, it is enough to show, by virtue of Lemma 2.14(a), that every filterbase $\mathcal{F}$ on $X$ $\theta_{3}$-adheres at some $x \in X$. Let $\mathcal{F}$ be a filterbase on $X$. Take $y_{0} \notin X$, and construct $Y = X \cup \{y_{0}\}$. Define, $\tau_{Y} = \{U \subseteq Y : y_{0} \notin U \} \cup \{U \subseteq Y : y_{0} \in U, F \subseteq U \text{ for some } F \in \mathcal{F}\}$. Then $\tau_{Y}$ is a topology on $Y$. Consider the function $\alpha : X \rightarrow Y$ by $\alpha(x) = x$. In order to avoid possible confusion, let us denote the closure and $\theta_{3}$-closure of a set $A$ in $X(Y)$, respectively, by $\cl_{X} A(\cl_{Y} A)$ and $\theta_{3}\cl_{X} A$ (respectively, $\theta_{3}\cl_{Y} A$). As $X$ is $\theta_{3}$-semiclosed in $X$, by the given condition, $S(\alpha, X) \equiv \cap \{\theta_{3}\cl_{Y} A(U) : U \in \mathrm{SO}(X)\} = \cap \{\theta_{3}\cl_{Y} U : U \in \mathrm{SO}(X)\} = \theta_{3}\cl_{Y} X$. We consider $y_{0} \in Y$ and $P_{0} \in \mathrm{SO}(y_{0})$. There is some $W \in \tau_{Y}$ such that $W \subseteq P_{0} \subseteq \cl_{Y} W$. If $y_{0} \notin W$, then $W \subseteq X$ and hence $\cl_{Y} W \cap X \neq \emptyset$. If on the other hand, $y_{0} \in W$, then there is some $F \in \mathcal{F}$ such that $F \subseteq W$, i.e., $\cl_{Y} F \subseteq \cl_{Y} W$. So, $X \cap \cl_{Y} W \neq \emptyset$.

So, in any case, $X \cap \cl_{Y} W \neq \emptyset$ and, consequently, as $\cl_{Y} W = \cl_{Y} P_{0}$, $X \cap \cl_{Y} P_{0} \neq \emptyset$. Thus, $y_{0} \in \theta_{3}\cl_{Y} X$. So, $y_{0} \in S(\alpha, x)$ for some $x \in X$. Consider any $V \in \mathrm{SO}(x)$ and $F \in \mathcal{F}$. Then $F \cup \{y_{0}\} \in \tau_{Y}$. Again, $Y - (F \cup \{y_{0}\})$ is a subset of $Y$ not containing $y_{0}$. Thus, $Y - (F \cup \{y_{0}\})$ is open in $Y$, which proves that $\cl_{Y} (F \cup \{y_{0}\}) = F \cup \{y_{0}\}$. Now, $\cl_{X} V \cap F = \alpha(\cl_{X} V) \cap \cl_{Y} (F \cup \{y_{0}\}) \neq \emptyset$. Thus, $x \in \theta_{3}$-ad $\mathcal{F}$.

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