AN EXAMPLE OF NONSYMMETRIC SEMI-CLASSICAL FORM OF CLASS $s = 1$; GENERALIZATION OF A CASE OF JACOBI SEQUENCE

MOHAMED JALEL ATIA

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Abstract. We give explicitly the recurrence coefficients of a nonsymmetric semi-classical sequence of polynomials of class $s = 1$. This sequence generalizes the Jacobi polynomial sequence, that is, we give a new orthogonal sequence $\{\hat{P}_n^{(\alpha,\alpha+1)}(x,\mu)\}$, where $\mu$ is an arbitrary parameter with $\Re(1-\mu) > 0$ in such a way that for $\mu = 0$ one has the well-known Jacobi polynomial sequence $\{\hat{P}_n^{(\alpha,\alpha+1)}(x)\}$, $n \geq 0$.

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1. Introduction. Many authors [1, 2, 3] have studied semi-classical sequences of polynomials of class $s = 1$. In particular, Bachène [2, page 87] gave the system fulfilled by such sequences using the structure relation and Belmehdi [3, page 272] gave the same system (in a more simple way) using directly the functional equation. This system is not linear and has not been sorted out before. The aim of this paper is to present a method that may give us some solutions.

In Section 2, we recall the general features which are needed in what follows. Section 3 is devoted to the setting of the problem, to give an integral representation and the expressions of the moments of the form $\hat{f}(\alpha,\alpha+1)(\mu)$ which generalizes the form $\hat{f}(\alpha,\alpha+1)$, where $\hat{f}(\alpha,\beta)$ is the Jacobi functional.

In Section 4, the recurrence coefficients of the semi-classical sequence of polynomials orthogonal with respect to $\hat{f}(\alpha,\alpha+1)(\mu)$ are explicitly given using the Laguerre-Freud equation of semi-classical orthogonal sequences of class $s = 1$ given in [3, page 272].

2. Preliminaries. Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and $\mathcal{P}'$ be its algebraic dual. We denote by $\langle u, f \rangle$ the action of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, we denote by $(u)_n := \langle u, x^n \rangle$, $n \geq 0$ the moments of $u$. Let us define the following operations on $\mathcal{P}'$:

- the left-multiplication of a linear functional by a polynomial
  \[
  \langle gu, f \rangle := \langle u, gf \rangle, \quad f, g \in \mathcal{P}, \ u \in \mathcal{P}',
  \]
  \[
  \text{ (2.1)}
  \]

- the derivative of a linear functional
  \[
  \langle u', f \rangle := -\langle u, f' \rangle, \quad f \in \mathcal{P}, \ u \in \mathcal{P}',
  \]
  \[
  \text{ (2.2)}
  \]
• the homothetic of a linear functional
\[ \langle h_a u, f \rangle := \langle u, h_a f \rangle, \quad a \in \mathbb{C} - \{0\}, \] (2.3)
where
\[ (h_a f)(x) = f(ax), \quad f \in \mathcal{P}, \ u \in \mathcal{P}', \] (2.4)
• the translation of a linear functional
\[ \langle \tau_b u, f \rangle := \langle u, \tau_{-b} f \rangle, \quad b \in \mathbb{C}, \] (2.5)
where
\[ (\tau_b f)(x) = f(x - b), \quad f \in \mathcal{P}, \ u \in \mathcal{P}', \] (2.6)
• the division of a linear functional by a polynomial of first degree
\[ \langle (x - c)^{-1} u, f \rangle := \langle u, \theta_c f \rangle, \quad c \in \mathbb{C}, \] (2.7)
where
\[ (\theta_c f)(x) = \frac{f(x) - f(c)}{x - c}, \quad f \in \mathcal{P}, \ u \in \mathcal{P}', \] (2.8)

using (2.1) and (2.2) we can easily prove
\[ (fu)' = f'u + fu', \quad f \in \mathcal{P}, \ u \in \mathcal{P}'. \] (2.9)

Definition 2.1 (see [4]). A sequence of polynomials \( \{\hat{P}_n\}_{n \geq 0} \) is said to be a monic orthogonal polynomial sequence with respect to the linear functional \( u \) if
(i) deg \( \hat{P}_n = n \) and the leading coefficient of \( \hat{P}_n(x) \) is equal to 1.
(ii) \( \langle u, \hat{P}_n \hat{P}_m \rangle = r_n \delta_{nm}, n, m \geq 0, r_n \neq 0, n \geq 0. \)

It is well known that a sequence of monic orthogonal polynomial satisfies a three-term recurrence relation
\[ \hat{P}_0(x) = 1, \quad \hat{P}_1(x) = x - \beta_0, \] \[ \hat{P}_{n+2}(x) = (x - \beta_{n+1}) \hat{P}_{n+1}(x) - y_{n+1} \hat{P}_n(x), \quad n \geq 0, \] (2.10)
with \( (\beta_n, y_{n+1}) \in \mathbb{C} \times \mathbb{C} - \{0\}, n \geq 0. \)

In such conditions, we say that \( u \) is regular or quasi-definite (see [4]). In what follows, we assume that the linear functionals are regular.
A shifting leaves invariant the orthogonality for the sequence \( \{\hat{P}_n\}_{n \geq 0} \). In fact, \( \hat{P}_n(x) = a^{-n} \hat{P}_n(ax + b), n \geq 0, \) fulfills the recurrence relation [6] and [8, page 265]
\[ \hat{P}_0(x) = 1, \quad \hat{P}_1(x) = x - \hat{\beta}_0, \] \[ \hat{P}_{n+2}(x) = (x - \hat{\beta}_{n+1}) \hat{P}_{n+1}(x) - \hat{y}_{n+1} \hat{P}_n(x), \quad n \geq 0 \] (2.11)
with \( \hat{\beta}_n = (\beta_n - b)/a, \hat{y}_{n+1} = (y_{n+1})/a^2, n \geq 0, a \in \mathbb{C} - \{0\}. \)
**Definition 2.2** (see [4]). \(\{\hat{P}_n\}_{n \geq 0}\) (respectively, the linear functional \(u\)) is semi-classical of class \(s\), if and only if the following statement holds: [6] and [7, pages 143–144].

There exist two polynomials \(\psi\) of degree \(p \geq 1\) and \(\phi\) of degree \(t \geq 0\), such that

\[
(\phi u)' + \psi u = 0,
\]

\[
\prod_{c \in Z_\phi} (|\psi(c) + \phi'(c)| + |\langle u, \theta_c(\psi) + \theta_2^c(\phi) \rangle|) \neq 0,
\]

where \(Z_\phi\) is the set of zeros of \(\phi\). The class of \(\{P_n\}_{n \geq 0}\) or \(u\) is given by \(s = \max(p - 1, t - 2)\) [7, pages 143–144].

If \(u\) is a semi-classical functional of class \(s\), then \(v = (h_{a-1} \circ \tau_{-b})u\) is also semi-classical of the same class and it verifies the equation \((\phi_1 v)' + \psi_1 v = 0\), where

\[
\phi_1(x) = a^{-t} \phi(ax + b), \quad \psi_1(x) = a^{1-t} \psi(ax + b).
\]

**3. Generalization of \(\mathcal{H}(\alpha, \alpha + 1)\) as a semi-classical sequence of class \(s = 1\)**

**3.1. Problem setting.** If \(u\) is a classical linear function, that is,

\[
(\phi(x) u)' + \psi(x) u = 0, \quad \deg \phi \leq 2, \quad \deg \psi = 1,
\]

from (2.9) the multiplication by \(x\) gives

\[
(x\phi(x) u)' - \phi(x) u + x\psi(x) u = 0, \quad \deg(x\phi) \leq 3, \quad \deg(x\psi - \phi) \leq 2.
\]

If we consider the following perturbed equation

\[
(x\phi(x) u(\mu))' + ((\mu - 1)\phi(x) + x\psi(x)) u(\mu) = 0,
\]

\[
\deg(x\phi) \leq 3, \quad \deg(x\psi + (\mu - 1)\phi) \leq 2,
\]

we obtain, under some conditions on \(\mu\), a linear functional \(u(\mu)\) of class \(s = 1\) which generalizes the classical linear functional \(u\).

**Examples**

(1) **The Hermite case.** One knows that the functional equation for the Hermite linear functional, noted \(\mathcal{H}\), is [6, page 117]

\[
\mathcal{H}' + 2x\mathcal{H} = 0
\]

multiplied by \(x\) gives

\[
(x\mathcal{H})' + (2x^2 - 1)\mathcal{H} = 0.
\]

Thus, we consider the functional equation

\[
(x\mathcal{H}(\mu))' + (2x^2 - 2\mu - 1)\mathcal{H}(\mu) = 0
\]
which is the functional equation of the well-known generalized-Hermite linear functional, noted \( \mathfrak{H}(\mu) \), which is regular for \( \mu \neq -n - 1/2, \ n \geq 0 \), and semi-classical of class \( s = 1 \) for \( \mu \neq 0 \) [4] and [5, page 243]. Notice that \( \mathfrak{H}(0) = \mathfrak{H} \).

(2) The Jacobi Case. Let us consider the functional equation for the Jacobi form, \( \mathcal{J}(\alpha, \beta) \):

\[
((x^2 - 1) \mathcal{J}(\alpha, \beta))' + (- (\alpha + \beta + 2)x + \alpha - \beta) \mathcal{J}(\alpha, \beta) = 0
\]

(3.7)
multiplication by \( x \) gives the following equation:

\[
((x^3 - x) \mathcal{J}(\alpha, \beta))' - (x^2 - 1) \mathcal{J}(\alpha, \beta) + (- (\alpha + \beta + 2)x^2 + (\alpha - \beta)x) \mathcal{J}(\alpha, \beta) = 0.
\]

(3.8)
Thus, consider

\[
((x^3 - x) \mathcal{J}(\alpha, \beta)(\mu))' + ((\mu - \alpha - \beta - 3)x^2 + (\alpha - \beta)x + 1 - \mu) \mathcal{J}(\alpha, \beta)(\mu) = 0.
\]

(3.9)
Notice that \( \mathcal{J}(\alpha, \beta)(0) = \mathcal{J}(\alpha, \beta) \).

(a) The Gegenbauer case (\( \alpha = \beta \)). In this case (3.9) becomes

\[
((x^3 - x) \mathcal{J}(\alpha, \alpha)(\mu))' + ((\mu - 2\alpha - 3)x^2 + 1 - \mu) \mathcal{J}(\alpha, \alpha)(\mu) = 0
\]

(3.10)
which is the functional equation of the symmetric semi-classical functional, regular for \( \mu \neq 2n + 2\alpha + 1, \mu \neq 2n + 1, \ n \geq 0 \), of class \( s = 1 \) for \( \mu \neq 0 \), and \( \mathcal{J}(\alpha, \alpha)(0) = \mathcal{J}(\alpha, \alpha) \).

In fact, in [1, page 317], we have

\[
((x^3 - x) u)' + 2(- (\tilde{\alpha} + \tilde{\beta} + 2)x^2 + \tilde{\beta} + 1) u = 0
\]

(3.11)
and if we denote by \( \{P_n\}_{n \geq 0} \) the sequence of monic polynomials orthogonal with respect to \( u \), then \( \{P_n\}_{n \geq 0} \) fulfills (2.10) such that

\[
\beta_n = 0,
\]

\[
Y_{2n+1} = \frac{(n + \tilde{\beta} + 1)(n + \tilde{\alpha} + \tilde{\beta} + 1)}{(2n + \tilde{\alpha} + \tilde{\beta} + 1)(2n + \tilde{\alpha} + \tilde{\beta} + 2)},
\]

(3.12)
\[
Y_{2n+2} = \frac{(n + 1)(n + \tilde{\alpha} + 1)}{(2n + \tilde{\alpha} + \tilde{\beta} + 2)(2n + \tilde{\alpha} + \tilde{\beta} + 3)},
\]

for \( n \geq 0 \). Put

\[
-2(\tilde{\alpha} + \tilde{\beta} + 2) = \mu - (2\alpha + 3), \quad 2(\tilde{\beta} + 1) = 1 - \mu,
\]

(3.13)
we obtain \( ((x^3 - x) u)' + ((\mu - 2\alpha - 3)x^2 + 1 - \mu) u = 0 \) with

\[
\beta_n = 0,
\]

\[
Y_{2n+1} = \frac{(2n + 2\alpha + 1 - \mu)(2n + 1 - \mu)}{(4n + 2\alpha + 1 - \mu)(4n + 2\alpha + 3 - \mu)},
\]

(3.14)
\[
Y_{2n+2} = \frac{4(n + 1)(n + \alpha + 1)}{(4n + 2\alpha + 3 - \mu)(4n + 2\alpha + 5 - \mu)},
\]

for \( n \geq 0 \).
(b) $\mathcal{J}(\alpha, \alpha + 1)$ case. If in (3.9), $\beta = \alpha + 1$ we get

$$
((x^3 - x)\mathcal{J}(\alpha, \alpha + 1)(\mu))' + ((\mu - 2\alpha - 4)x^2 - x + 1 - \mu)\mathcal{J}(\alpha, \alpha + 1)(\mu) = 0. \quad (3.15)
$$

In what follows, we will look for the regular linear functional, $\mathcal{J}(\alpha, \alpha + 1)(\mu)$ which is a solution of (3.15) and we denote by $\{P_n\}_{n \geq 0}$ the sequence of monic orthogonal polynomials with respect to $\mathcal{J}(\alpha, \alpha + 1)(\mu)$ and by $\beta_n, \gamma_n, n \geq 0$ the recurrence coefficients of $P_n$.

**Remark 3.1.** The solutions of the functional equation (3.15) depend on the value of $(\mathcal{J}(\alpha, \alpha + 1)(\mu))_1 = \beta_0$, in fact,

$$
(((x^3 - x)\mathcal{J}(\alpha, \alpha + 1)(\mu))' + ((\mu - 2\alpha - 4)x^2 - x + 1 - \mu)\mathcal{J}(\alpha, \alpha + 1)(\mu), 1) = 0, \quad (3.16)
$$

then, using (2.2), one has

$$
(((\mu - 2\alpha - 4)x^2 - x + 1 - \mu)\mathcal{J}(\alpha, \alpha + 1)(\mu), 1) = (\mu - 2\alpha - 4)(\mathcal{J}(\alpha, \alpha + 1)(\mu))_2 - (\mathcal{J}(\alpha, \alpha + 1)(\mu))_1 + 1 - \mu = 0,
$$

but $(\mathcal{J}(\alpha, \alpha + 1)(\mu))_2 = \gamma_1 + \beta_0^2$ and $(\mathcal{J}(\alpha, \alpha + 1)(\mu))_1 = \beta_0$ then

$$
(\mu - 2\alpha - 4)\gamma_1 + (\mu - 2\alpha - 4)\beta_0^2 - \beta_0 + 1 - \mu = 0. \quad (3.18)
$$

First we search an integral representation in order to obtain $\beta_0$.

**3.2 An integral representation**

**Proposition 3.2.** An integral representation of a linear functional $\mathcal{J}(\alpha, \alpha + 1)(\mu)$ is

$$
\langle \mathcal{J}(\alpha, \alpha + 1)(\mu), f(x) \rangle = \Gamma((2\alpha + 3 - \mu)/2) \Gamma(1 - \mu)/2 \Gamma(1 + \alpha) \int_{-1}^{+1} |x|^{-\mu} (1 - x^2)^{\alpha}(1 - x)f(x) \, dx
$$

with $\text{Re}(1 - \mu) > 0$, that is, $\text{Re}(\alpha + 1) > 0$.

**Proof.** A solution of (3.15) has the integral representation

$$
\langle \mathcal{J}(\alpha, \alpha + 1)(\mu), f(x) \rangle = \int_C U(x)f(x) \, dx, \quad f \in \mathcal{P}
$$

if the following conditions hold [5]:

$$
((x^3 - x)U(x))' + ((\mu - 2\alpha - 4)x^2 - x + 1 - \mu)U(x) = 0,
$$

$$
(x^3 - x)U(x)f(x) \bigg|_C = 0, \quad f \in \mathcal{P},
$$

where $C$ is an acceptable integration path. We solve the first condition as a differential equation:

$$
((x^3 - x)U(x))' + ((\mu - 2\alpha - 4)x^2 - x + 1 - \mu)U(x) = 0 \quad (3.22)
$$
or, equivalently,
\[(x^3 - x)U'(x) + ((\mu - 2\alpha - 1)x^2 - x - \mu)U(x) = 0,
\]
\[
\frac{U'(x)}{U(x)} = -\frac{(\mu - 2\alpha - 1)x^2 - x - \mu}{x^3 - x} = -\frac{(\mu - 2\alpha - 1)x^2 - x - \mu}{x(x-1)(x+1)}.
\]

Thus
\[
\frac{U'(x)}{U(x)} = -\frac{\mu x + (\alpha + 1)}{x(x-1)(x+1)}
\]

and
\[
U(x) = \begin{cases} 
  k|x|^{-\mu}(1-x^2)^\alpha(1-x), & |x| < 1, \\
  0, & |x| > 1.
\end{cases}
\]

If we assume \(\text{Re}(1-\mu) > 0, \text{Re}(\alpha + 1) > 0\), then
\[
(x^3 - x)U(x)f(x) = k(x^3 - x)|x|^{-\mu}(1-x^2)^\alpha(1-x)f(x)|_{-1}^{+1} = 0
\]
holds.

**Determination of the normalisation factor.**

\[
\langle \mathcal{J}(\alpha, \alpha + 1)(\mu), 1 \rangle = k_1 \int_{-1}^{+1} |x|^{-\mu}(1-x^2)^\alpha(1-x) \, dx
\]

\[
= k_1 \int_{-1}^{+1} |x|^{-\mu}(1-x^2)^\alpha \, dx
\]

\[
= 2k_1 \int_{0}^{+1} (x)^{-\mu}(1-x^2)^\alpha \, dx
\]

\[
= 2k_1 \frac{1}{2} B \left( \frac{1-\mu}{2}, \alpha + 1 \right) = 1,
\]

where \(B(p,q)\) is the beta function. Thus, from
\[
\langle \mathcal{J}(\alpha, \alpha + 1)(\mu), 1 \rangle = k_1 B \left( \frac{1-\mu}{2}, \alpha + 1 \right) = k_1 \frac{\Gamma((1-\mu)/2)\Gamma(1+\alpha)}{\Gamma((2\alpha + 3 - \mu)/2)} = 1
\]

we get
\[
k_1 = \frac{\Gamma((2\alpha + 3 - \mu)/2)}{\Gamma((1-\mu)/2)\Gamma(1+\alpha)}.
\]

Conversely, using this integral representation, we give explicitly the expressions of the moments and the functional equation (3.15).

**Lemma 3.3.** Using the integral representation we have
\[
\langle \mathcal{J}(\alpha, \alpha + 1)(\mu), 1 \rangle = k_1 \frac{\Gamma((1-\mu)/2)\Gamma(1+\alpha)}{\Gamma((2\alpha + 3 - \mu)/2)} = 1
\]

we get
\[
k_1 = \frac{\Gamma((2\alpha + 3 - \mu)/2)}{\Gamma((1-\mu)/2)\Gamma(1+\alpha)}.
\]

Conversely, using this integral representation, we give explicitly the expressions of the moments and the functional equation (3.15).

**3.3. The expressions of the moments.** Using the integral representation we have a relation between \(\langle \mathcal{J}(\alpha, \alpha + 1)(\mu) \rangle_{2n+1} \) and \(\langle \mathcal{J}(\alpha, \alpha + 1)(\mu) \rangle_{2n+2} \) and a relation between \(\langle \mathcal{J}(\alpha, \alpha + 1)(\mu) \rangle_{2n+2} \) and \(\langle \mathcal{J}(\alpha, \alpha + 1)(\mu) \rangle_{2n} \). Then, using these two relations, we obtain the functional equation.

**Lemma 3.3.** Using the integral representation we have
\[
\langle \mathcal{J}(\alpha, \alpha + 1)(\mu) \rangle_{2n+1} = -\langle \mathcal{J}(\alpha, \alpha + 1)(\mu) \rangle_{2n+2}, \quad n \geq 0.
\]
Using (3.30) and (3.33) we can find the functional equation (3.15). 

Proof.

\[
\langle \chi(\alpha, \alpha + 1)(\mu), x^{2n+1} + x^{2n+2} \rangle = k_1 \int_{-1}^{1} |x|^{-\mu} (1-x^2)^{\alpha} (1-x) (x^{2n+1} + x^{2n+2}) \, dx \\
= k_1 \int_{-1}^{1} x^{2n+1} |x|^{-\mu} (1-x^2)^{\alpha+1} \, dx = 0
\]

(3.31)

because \(x^{2n+1} |x|^{-\mu} (1-x^2)^{\alpha+1}\) is an odd function.

\[\square\]

Lemma 3.4. Using the integral representation we have

\[
\left( \langle \chi(\alpha, \alpha + 1)(\mu) \rangle \right)_{2n+2} = \frac{\Gamma((2n+3-\mu)/2)\Gamma(\alpha+1)}{\Gamma((2n+2\alpha+5-\mu)/2)}
\]

(3.32)

and, in particular,

\[
(2n+2\alpha+3-\mu) \left( \langle \chi(\alpha, \alpha + 1)(\mu) \rangle \right)_{2n+2} = (2n+1-\mu) \left( \langle \chi(\alpha, \alpha + 1)(\mu) \rangle \right)_{2n}, \quad n \geq 0.
\]

(3.33)

Proof. From

\[
\langle \chi(\alpha, \alpha + 1)(\mu), x^{2n+2} \rangle = k_1 \int_{-1}^{1} |x|^{-\mu} (1-x^2)^{\alpha} (1-x) x^{2n+2} \, dx \\
= k_1 \int_{-1}^{1} x^{2n+2} |x|^{-\mu} (1-x^2)^{\alpha} \, dx
\]

taking into account that \(x^{2n+3} |x|^{-\mu} (1-x^2)^{\alpha}\) is an odd function,

\[
\langle \chi(\alpha, \alpha + 1)(\mu), x^{2n+2} \rangle = 2k_1 \int_{0}^{1} x^{2n+2-\mu} (1-x^2)^{\alpha} \, dx \\
= 2k_1 \frac{1}{2} B \left( \frac{2n+3-\mu}{2}, \alpha+1 \right),
\]

(3.34)

where \(B(p,q)\) is the beta function

\[
\langle \chi(\alpha, \alpha + 1)(\mu), x^{2n+2} \rangle = \frac{\Gamma((2n+3-\mu)/2)\Gamma(\alpha+1)}{\Gamma((2n+2\alpha+5-\mu)/2)} \\
= \frac{2n+1-\mu}{2n+2\alpha+3-\mu} \langle \chi(\alpha, \alpha + 1)(\mu), x^{2n} \rangle, \quad n \geq 0.
\]

(3.35)

Using (3.30) and (3.33) we can find the functional equation (3.15).

From (3.33), we have, for \(n \geq 0\),

\[
(2n+2\alpha+3-\mu) \left( \langle \chi(\alpha, \alpha + 1)(\mu) \rangle \right)_{2n+2} = (2n+1-\mu) \left( \langle \chi(\alpha, \alpha + 1)(\mu) \rangle \right)_{2n},
\]

(3.37)

with (3.30), one has

\[
(2n+2\alpha+4-\mu) \left( \langle \chi(\alpha, \alpha + 1)(\mu) \rangle \right)_{2n+2} \\
= - \left( \langle \chi(\alpha, \alpha + 1)(\mu) \rangle \right)_{2n+1} + (2n+1-\mu) \left( \langle \chi(\alpha, \alpha + 1)(\mu) \rangle \right)_{2n}, \quad n \geq 0.
\]

(3.38)
Using (2.1) and (2.2), we get, for \( n \geq 0 \),
\[
\langle ((x^3 - x) \tilde{f}(\alpha, \alpha+1)(\mu))' + ((\mu - 2\alpha - 4)x^2 - x - (\mu - 1)) \tilde{f}(\alpha, \alpha+1)(\mu), x^{2n} \rangle = 0. 
\tag{3.39}
\]
From (3.15) and (3.33), we have, for \( n \geq 0 \),
\[
(2n + 2\alpha + 5 - \mu) (\tilde{f}(\alpha, \alpha+1)(\mu))_{2n+3} = (2n + 3 - \mu) (\tilde{f}(\alpha, \alpha+1)(\mu))_{2n+1}. 
\tag{3.40}
\]
Thus, taking into account (3.30), one has
\[
(2n + 2\alpha + 5 - \mu) (\tilde{f}(\alpha, \alpha+1)(\mu))_{2n+3} = -(\tilde{f}(\alpha, \alpha+1)(\mu))_{2n+2} + (2n + 2 - \mu) (\tilde{f}(\alpha, \alpha+1)(\mu))_{2n+1}, \quad n \geq 0. 
\tag{3.41}
\]
From
\[
\langle ((x^3 - x) \tilde{f}(\alpha, \alpha+1)(\mu))' + ((\mu - 2\alpha - 4)x^2 - x - (\mu - 1)) \tilde{f}(\alpha, \alpha+1)(\mu), x^{2n+1} \rangle = 0, \quad n \geq 0.
\tag{3.42}
\]
equations (3.39) and (3.42) give
\[
\langle ((x^3 - x) \tilde{f}(\alpha, \alpha+1)(\mu))' + ((\mu - 2\alpha - 4)x^2 - x - (\mu - 1)) \tilde{f}(\alpha, \alpha+1)(\mu), x^n \rangle = 0, \quad n \geq 0. 
\tag{3.43}
\]
Hence
\[
((x^3 - x) \tilde{f}(\alpha, \alpha+1)(\mu))' + ((\mu - 2\alpha - 4)x^2 - x - (\mu - 1)) \tilde{f}(\alpha, \alpha+1)(\mu) = 0. 
\tag{3.44}
\]

**Corollary 3.5.** From (3.30) and (3.33) we deduce the expressions of the moments:
\[
(\tilde{f}(\alpha, \alpha+1)(\mu))_{2n+1} = -\prod_{i=0}^{n} \frac{(2i + 1 - \mu)}{(2\alpha + 2i + 3 - \mu)}, \quad n \geq 0,
\tag{3.45}
\]
\[
(\tilde{f}(\alpha, \alpha+1)(\mu))_{2n+2} = -(\tilde{f}(\alpha, \alpha+1)(\mu))_{2n+1}, \quad n \geq 0.
\]

4. The recurrence coefficients \( \beta_n, \gamma_n, n \geq 0 \)

4.1. The system satisfied by recurrence coefficients of semi-classical sequences of class \( s = 1 \). Assuming that \( u \) is semi-classical of class \( s = 1 \), then \( u \) satisfies
\[
(\phi u)' + \psi u = 0 
\tag{4.1}
\]
with
\[
\phi(x) = \sum_{k=0}^{3} c_k x^k, \quad \sum_{k=0}^{3} |c_k| \neq 0, \quad \psi(x) = \sum_{k=0}^{2} a_k x^k, \quad |a_2| + |a_1| \neq 0 \tag{4.2}
\]
(see [3, page 272]). Furthermore, the nonlinear system satisfied by the recurrence
coefficients of semi-classical orthogonal sequences of class $s = 1$ is

\[(a_2 - 2nc_3)(y_n + y_{n+1}) = 4c_3 \sum_{k=1}^{n-1} y_k + 2 \sum_{k=0}^{n-1} (\theta_{\beta_n} \phi)(\beta_k) - \psi(\beta_n), \quad n \geq 2,
\]

\[(a_2 - 2c_3)(y_1 + y_2) = 2(\theta_{\beta_1} \phi)(\beta_0) - \psi(\beta_1), \quad (4.3)
\]

\[a_2 y_1 = -\psi(\beta_0),
\]

\[(a_2 - (2n + 1)c_3) y_{n+1} \beta_{n+1} = \sum_{k=0}^{n} \phi(\beta_k) + c_3 \left( 2y_n \left( n\beta_n + \sum_{k=0}^{n} \beta_k + 3 \sum_{k=1}^{n} y_k (\beta_k + \beta_{k-1}) \right) \right)
\]

\[+ c_2 \left( (2n + 1) y_{n+1} + 2 \sum_{k=1}^{n} y_k \right) - (a_2 \beta_n + a_1) y_{n+1}, \quad n \geq 1,
\]

\[(4.4)
\]

In our case, since $c_3 = -c_1 = 1$, $c_2 = c_0 = 0$, the first equation of (4.3) becomes

\[(\mu - 2n - 2\alpha - 4)(y_n + y_{n+1}) = 4 \sum_{k=1}^{n-1} y_k + 2 \sum_{k=0}^{n-1} (\theta_{\beta_n} \phi)(\beta_k) - \psi(\beta_n), \quad n \geq 2. \quad (4.5)
\]

Using (2.7), we get

\[\begin{align*}
(\mu - 2n - 2\alpha - 4)y_{n+1} &= -(\mu - 2n - 2\alpha - 4)y_n + 4 \sum_{k=1}^{n-1} y_k + 2 \sum_{k=0}^{n-1} (\beta_n^2 + \beta_k^2 + \beta_n \beta_k - 1) \\
&\quad - (\mu - 2\alpha - 4)\beta_n^2 + \beta_n - (1 - \mu) \\
&\quad = -(\mu - 2n - 2\alpha - 4)y_n + 4 \sum_{k=1}^{n-1} y_k + 2 \sum_{k=0}^{n-1} \beta_k^2 + 2\beta_n \sum_{k=0}^{n-1} \beta_k \\
&\quad + (2n + 2\alpha + 4 - \mu)\beta_n^2 + \beta_n + \mu - 2n - 1, \quad n \geq 2
\end{align*}
\]

\[(4.6)
\]

then

\[\begin{align*}
(\mu - 2n - 2\alpha - 6)y_{n+2} &= -(\mu - 2n - 2\alpha - 6)y_{n+1} + 4 \sum_{k=1}^{n} y_k + 2 \sum_{k=0}^{n} \beta_k^2 + 2\beta_{n+1} \sum_{k=0}^{n} \beta_k \\
&\quad + (2n + 2\alpha + 6 - \mu)\beta_{n+1}^2 + \beta_{n+1} + \mu - 2n - 3, \quad n \geq 1
\end{align*}
\]

\[(4.7)
\]

If we subtract both identities,

\[\begin{align*}
(\mu - 2n - 2\alpha - 6)y_{n+2} &= -(\mu - 2n - 2\alpha - 6)y_{n+1} + (\mu - 2n - 2\alpha - 4)y_{n+1} \\
&\quad + (\mu - 2n - 2\alpha - 4)y_n + 2\beta_n^2 \\
&\quad + 2\beta_{n+1} \sum_{k=0}^{n} \beta_k - 2\beta_n \sum_{k=0}^{n-1} \beta_k + (2n + 2\alpha + 6 - \mu)\beta_{n+1}^2 \\
&\quad - (2n + 2\alpha + 4 - \mu)\beta_n^2 + \beta_{n+1} - \beta_n - 2, \quad n \geq 1
\end{align*}
\]

\[(4.8)
\]
Thus the first equation of (4.3) becomes
\[
(\mu - 2n - 2\alpha - 6)y_{n+2} = 2y_{n+1} + (\mu - 2n - 2\alpha)y_n + 2\beta_{n+1} \sum_{k=0}^{n} \beta_k - 2\beta_n \sum_{k=0}^{n-1} \beta_k \\
+ (2n + 2\alpha + 6 - \mu)\beta_{n+1}^2 - (2n + 2\alpha + 2 - \mu)\beta_n^2 \\
+ (\beta_{n+1} - \beta_n) - 2, \quad n \geq 1.
\]
(4.9)

On the other hand, (4.4) becomes
\[
(\mu - 2n - 2\alpha - 5)y_{n+1} \beta_{n+1} = \sum_{k=0}^{n} \phi(\beta_k) + \left( 2y_n \left( n\beta_n + \sum_{k=0}^{n} \beta_k \right) + 3 \sum_{k=1}^{n} y_k (\beta_k + \beta_{k-1}) \right) \\
+ c_2 \left( 2(n+1)y_{n+1} + 2 \sum_{k=1}^{n} y_k \right) - ((\mu - 2\alpha - 4)\beta_n - 1)y_{n+1} \\
= \sum_{k=0}^{n} (\beta_k^3 - \beta_k) + \left( 2y_n \left( n\beta_n + \sum_{k=0}^{n} \beta_k \right) + 3 \sum_{k=1}^{n} y_k (\beta_k + \beta_{k-1}) \right) \\
- ((\mu - 2\alpha - 4)\beta_n - 1)y_{n+1}, \quad n \geq 1.
\]
(4.10)

Shifting the indices and subtracting, we get
\[
(\mu - 2n - 2\alpha - 7)y_{n+2} \beta_{n+2} = (\mu - 2n - 2\alpha - 5)y_{n+1} \beta_{n+1} \\
+ \beta_{n+1}^3 - \beta_{n+1} + 3y_{n+1} (\beta_{n+1} + \beta_n) \\
+ \left( 2y_{n+2} \left( (n+1)\beta_{n+1} + \sum_{k=0}^{n} \beta_k \right) \right) \\
- \left( 2y_{n+1} \left( n\beta_n + \sum_{k=0}^{n} \beta_k \right) \right) - ((\mu - 2\alpha - 4)\beta_{n+1} - 1)y_{n+2} \\
+ ((\mu - 2\alpha - 4)\beta_n - 1)y_{n+1}, \quad n \geq 0.
\]
(4.11)

Thus, from (4.9) and (4.11) we have the following.

**Proposition 4.1.**

\( (\mu - 2n - 2\alpha - 6)y_{n+2} \)
\[
= 2y_{n+1} + (\mu - 2n - 2\alpha)y_n + 2\beta_{n+1} \sum_{k=0}^{n} \beta_k - 2\beta_n \sum_{k=0}^{n-1} \beta_k \\
+ (2n + 2\alpha + 6 - \mu)\beta_{n+1}^2 - (2n + 2\alpha + 2 - \mu)\beta_n^2 + (\beta_{n+1} - \beta_n) - 2, \quad n \geq 1
\]
(4.12)

\( (\mu - 2n - 2\alpha - 6)(y_1 + y_2) = 2(\beta_2^2 + \beta_0 \beta_1 + \beta_0^2 - 1) - (\mu - 2\alpha - 4)\beta_1^2 + \beta_1 - (1 - \mu) \)
(4.13)

\( (\mu - 2n - 2\alpha - 4)y_1 = - (\mu - 2\alpha - 4)\beta_0^2 + \beta_0 - (1 - \mu). \)
(4.14)

\( (\mu - 2n - 2\alpha - 7)y_{n+2} \beta_{n+2} \)
\[
= \beta_{n+1}^3 - \beta_{n+1} + (2n + 2\alpha + 8 - \mu)\beta_{n+2} \beta_{n+1} + (\mu - 2n - 2\alpha - 2)y_{n+1} \beta_{n+1} \\
+ (\mu - 2n - 2\alpha - 1)y_{n+1} \beta_n + \left( 2 \sum_{k=0}^{n} \beta_k + 1 \right) (y_{n+2} - y_{n+1}), \quad n \geq 0
\]
(4.15)

\( (\mu - 2n - 2\alpha - 5)y_1 \beta_1 = \beta_0^3 - \beta_0 + y_1 (2\beta_0 - (\mu - 2\alpha - 4)\beta_0 + 1). \)
(4.16)
Next, we will find the expressions of the recurrence parameters $\beta_n, \gamma_n$, $n \geq 0$.

Since $\beta_0 = -(\mu - 1)/(\mu - 2\alpha - 3)$ and from (4.14) we have

$$
\gamma_1 = \frac{-(\mu - 2\alpha - 4)\beta_0^2 + \beta_0 - (1 - \mu)}{\mu - 2\alpha - 4} = -\beta_0^2 + \frac{\beta_0}{\mu - 2\alpha - 4} + \frac{\mu - 1}{\mu - 2\alpha - 4}
$$

(4.17)

Using (4.16), (4.17) gives

$$
\beta_1 = \frac{\beta_0^3 - \beta_0 + \gamma_1(- (\mu - 2\alpha - 6)\beta_0 + 1)}{(\mu - 2\alpha - 5)\gamma_1} = \frac{\mu(\mu - 2\alpha - 4) - (2\alpha + 1)}{(2\alpha + 3)(2\alpha + 5 - \mu)}. \quad (4.18)
$$

With $\beta_0, \beta_1, \gamma_1, (4.13)$ gives

$$
\gamma_2 = -\gamma_1 + \frac{2(\beta_1^2 + \beta_0\beta_1 + \beta_0^2 - 1) - (\mu - 2\alpha - 4)\beta_0^2 + \beta_0 - (1 - \mu)}{\mu - 2\alpha - 6} = \frac{2(2\alpha + 3 - \mu)}{(2\alpha + 5 - \mu)^2}. \quad (4.19)
$$

With $\beta_0, \beta_1, \gamma_1, \gamma_2, (4.15)$ and some easy computations

$$
\beta_2 = \frac{-\mu(\mu - 2\alpha - 6) + (2\alpha + 1)}{(2\alpha + 5 - \mu)(2\alpha + 7 - \mu)}. \quad (4.20)
$$

**PROPOSITION 4.2.** Assuming

$$
\beta_0 = -\frac{\mu - 1}{\mu - 2\alpha - 3},
$$

$$
\beta_{n+1} = (-1)^n \frac{\mu(\mu - 2n - 2\alpha - 4) + (-1)^n(2\alpha + 1)}{(2n + 2\alpha + 3 - \mu)(2n + 2\alpha + 5 - \mu)}, \quad (4.21)
$$

$$
\gamma_{2n+1} = \frac{2(n + \alpha + 1)(2n + 1 - \mu)}{(4n + 2\alpha + 3 - \mu)^2},
$$

$$
\gamma_{2n+2} = \frac{(2n + 2)(2n + 2\alpha + 3 - \mu)}{(4n + 2\alpha + 5 - \mu)^2},
$$

for $n \geq 0$ and assume $\mu \neq 2n + 1$, $\mu \neq 2n + 2\alpha + 1$, $\alpha \neq -n - 1$, $n \geq 0$.

**LEMMA 4.3.** If $E_n = \sum_{k=0}^{n} \beta_k$, $n \geq 0$, then

$$
E_{2n} = -\left(\frac{2n + 1 - \mu}{4n + 2\alpha + 3 - \mu}\right), \quad E_{2n+1} = -\frac{2n + 2}{4n + 2\alpha + 5 - \mu}, \quad n \geq 0. \quad (4.22)
$$
**Proof.** \( E_0 = \beta_0 \). For \( n \geq 0 \), we have

\[
E_{2n+1} = \sum_{k=0}^{n} (\beta_{2k} + \beta_{2k+1})
\]

\[
= \sum_{k=0}^{n} \frac{\mu(\mu - 4k - 2\alpha - 2) + 2\alpha + 1}{(4k + 2\alpha + 1 - \mu)(4k + 2\alpha + 3 - \mu)} + \frac{\mu(\mu - 4k - 2\alpha - 4) + 2\alpha - 1}{(4k + 2\alpha + 3 - \mu)(4k + 2\alpha + 5 - \mu)}
\]

\[
= \sum_{k=0}^{n} \left( \frac{1}{2} \frac{\mu - 2\alpha - 1}{(-4k - 2\alpha - 1 + \mu)} - \frac{1}{2} \frac{\mu + 2\alpha + 1}{(-4k - 2\alpha - 3 + \mu)} \right)
\]

\[
= \frac{\mu - 2\alpha - 1}{2} \left( -\frac{1}{-2\alpha - 1 + \mu} - \frac{1}{-4n - 2\alpha - 5 + \mu} \right)
\]

\[
= \frac{\mu - 2\alpha - 1}{2} \left( \frac{-4n - 4}{(-2\alpha - 1 + \mu)(-4n - 2\alpha - 5 + \mu)} \right)
\]

\[
E_{2n+1} = \frac{2n + 2}{(4n + 2\alpha + 5 - \mu)}, \quad \mu \neq 4n + 2\alpha + 5, \quad n \geq 0. \tag{4.24}
\]

Calculus of

\[
E_{2n+2} = E_{2n+1} + \beta_{2n+2}
\]

\[
= \frac{2n + 2}{4n + 2\alpha + 5 - \mu} - \frac{\mu(\mu - 4n - 2\alpha - 6) + 2\alpha + 1}{(4n + 2\alpha + 5 - \mu)(4n + 2\alpha + 7 - \mu)}
\]

\[
= - \frac{1}{4n + 2\alpha + 5 - \mu} \left( 2n + 2 + \frac{\mu(\mu - 2\alpha - 4n - 6) + 2\alpha + 1}{4n + 2\alpha + 7 - \mu} \right)
\]

\[
= - \frac{1}{4n + 2\alpha + 5 - \mu} \left( \frac{\mu^2 - (6n + 2\alpha + 8)\mu + (2n + 2)(4n + 2\alpha + 7) + 2\alpha + 1}{4n + 2\alpha + 7 - \mu} \right)
\]

\[
= - \frac{1}{4n + 2\alpha + 5 - \mu} \left( \frac{(4n + 2\alpha + 5 - \mu)(2n + 3 - \mu)}{4n + 2\alpha + 7 - \mu} \right), \tag{4.25}
\]

\[
E_{2n+2} = -\frac{2n + 3 - \mu}{4n + 2\alpha + 7 - \mu}, \quad \mu \neq 4n + 2\alpha + 7, \quad n \geq 0. \tag{4.26}
\]
**Proof of Proposition 4.2.** Suppose that we have

\[
\beta_0 = -\frac{\mu - 1}{\mu - 2\alpha - 3},
\]

\[
\beta_{2k+1} = \frac{\mu(\mu - 4k - 2\alpha - 4) - (2\alpha + 1)}{(4k + 2\alpha + 3 - \mu)(4k + 2\alpha + 5 - \mu)}, \quad 0 \leq k \leq n,
\]

\[
\beta_{2k} = -\frac{\mu(\mu - 4k - 2\alpha - 2) + (2\alpha + 1)}{(4k + 2\alpha + 1 - \mu)(4k + 2\alpha + 3 - \mu)}, \quad 1 \leq k \leq n,
\]

\[
y_{2k+1} = \frac{(4n - \mu + 2\alpha + 6)\beta_{2n+1}^2 - (4n - \mu + 2\alpha + 2)\beta_{2n}^2}{(4k + 2\alpha + 3 - \mu)^2}, \quad 0 \leq k \leq n - 1,
\]

\[
y_{2k+2} = \frac{(2k + 2)(2k + 2\alpha + 3 - \mu)}{(4k + 2\alpha + 5 - \mu)^2}, \quad 0 \leq k \leq n - 1,
\]

and, using (4.10), (4.13), we prove by induction \(\beta_{2n+2}, \beta_{2n+3}, y_{2n+2}, \) and \(y_{2n+3} \). The substitution \(n \to 2n\) in (4.10) gives

\[
(\mu - 2\alpha - 4n - 6)y_{2n+2} = 2y_{2n+1} + (\mu - 2\alpha - 4n)y_{2n+2} + 2\beta_{2n+1}E_{2n} - 2\beta_{2n}E_{2n-1} - (4n - \mu + 2\alpha + 6)\beta_{2n+1}^2 - (4n - \mu + 2\alpha + 2)\beta_{2n}^2
\]

\[
+ (\beta_{2n+1} - \beta_{2n}) - 2, \quad n \geq 1.
\]

We suppose known \(y_{2n+1}, y_{2n}, \beta_{2n+1}, \beta_{2n}, E_{2n}, \) and \(E_{2n-1}\) and then we evaluate \(y_{2n+2}\) for the proof by recurrence; because of cumbersome computation, using Maple. The substitution \(n \to 2n + 1\) in (4.10) gives (see appendix)

\[
(\mu - 2\alpha - 4n - 8)y_{2n+3} = 2y_{2n+2} + (\mu - 2\alpha - 4n - 2)y_{2n+1} + 2\beta_{2n+2}E_{2n+1} - 2\beta_{2n+1}E_{2n} - (4n - \mu + 2\alpha + 8)\beta_{2n+2}^2 - (4n - \mu + 2\alpha + 4)\beta_{2n+1}^2
\]

\[
+ (\beta_{2n+2} - \beta_{2n+1}) - 2, \quad n \geq 0.
\]

The substitution \(n \to 2n + 1\) in (4.13) gives (see appendix)

\[
(\mu - 2\alpha - 4n - 7)y_{2n+2}\beta_{2n+2} = \beta_{2n+1}^3 - \beta_{2n+1} + (-\mu + 2\alpha + 4n + 5)\beta_{2n+1}y_{2n+2}
\]

\[
- (-\mu + 2\alpha + 4n + 2)\beta_{2n+1}y_{2n+1} - (-\mu + 2\alpha + 4n + 1)\beta_{2n}y_{2n+1}
\]

\[
+ (2E_{2n+1})(y_{2n+2} - y_{2n+1}), \quad n \geq 0.
\]

Finally, the substitution \(n \to 2n + 2\) in (4.13) gives (see appendix)

\[
(\mu - 2\alpha - 4n - 5)y_{2n+3}\beta_{2n+3} = \beta_{2n+2}^3 - \beta_{2n+2} + (-\mu + 2\alpha + 4n + 10)\beta_{2n+2}y_{2n+3}
\]

\[
- (-\mu + 2\alpha + 4n + 4)\beta_{2n+2}y_{2n+2} - (-\mu + 2\alpha + 4n + 3)\beta_{2n+1}y_{2n+2}
\]

\[
+ (2E_{2n+1} + 1)(y_{2n+3} - y_{2n+2}), \quad n \geq 0.
\]
**Remarks.** (1) An homothetie of rapport \(-1\) gives a generalization of \(f(\alpha + 1, \alpha)\), with (2.11), (2.13) we have

\[
((x^3 - x)u)' + ((\mu - 2\alpha - 4)x^2 + x - (\mu - 1))u = 0,
\]

(4.32)

\[
\beta_0 = \frac{\mu - 1}{\mu - 2\alpha - 3},
\]

\[
\beta_{n+1} = (-1)^n \frac{\mu(\mu - 2n - 2\alpha - 4) + (-1)^n(2\alpha + 1)}{(2n + 2\alpha + 3 - \mu)(2n + 2\alpha + 5 - \mu)},
\]

(4.33)

\[
y_{2n+1} = 2 \frac{(n + \alpha + 1)(2n + 1 - \mu)}{(4n + 2\alpha + 3 - \mu)^2},
\]

\[
y_{2n+2} = \frac{(2n + 2)(2n + 2\alpha + 3 - \mu)}{(4n + 2\alpha + 5 - \mu)^2},
\]

for \(n \geq 0\).

(2) For \(\mu = 2\alpha + 4\), we have an apparent particular case

\[
((x^3 - x)u)' + (x - (2\alpha + 3))u = 0,
\]

(4.34)

\[
\beta_0 = -(2\alpha + 3),
\]

\[
\beta_{n+1} = (-1)^n \frac{(2\alpha + 4)(-2n) + (-1)^n(2\alpha + 1)}{(2n - 1)(2n + 1)},
\]

(4.35)

\[
y_{2n+1} = 2 \frac{(n + \alpha + 1)(2n - 2\alpha - 3)}{(4n - 1)^2},
\]

\[
y_{2n+2} = \frac{(2n + 2)(2n - 1)}{(4n + 1)^2},
\]

for \(n \geq 0\).

5. Appendix. In this appendix, we give both the input and output of the Maple programme used to carry out the computations of Section 4.

```maple
> restart;
> beta0:=-(-mu-1)/(mu-2*alpha-3);

\[
\beta_0 := -\frac{\mu - 1}{\mu - 2\alpha - 3}
\]

> gamma1:=factor(simplify(1/(mu-2*alpha-4)*((2*alpha+4-mu)*beta0^2 +beta0+mu-1)));

\[
\gamma_1 := -2 \frac{(1 + \alpha)(\mu - 1)}{(\mu - 2\alpha - 3)^2}
\]

> beta1:=collect(factor(simplify(1/((mu-2*alpha-5)*gamma1)*(beta0^3 -beta0+gamma1*-((mu-2*alpha-6)*beta0+1)))),mu);

\[
E_1 := \text{collect(simplify(beta0+beta1),mu)};
\]

\[
\beta_1 := \frac{\mu^2 + (-2\alpha - 4)\mu - 2\alpha - 1}{(\mu - 2\alpha - 3)(\mu - 2\alpha - 5)}
\]

\[
E_1 := \frac{2}{\mu - 2\alpha - 5}
\]
```
\( \gamma_2 := -\frac{2 \mu - 2 \alpha - 3}{(\mu - 2 \alpha - 5)^2} \)

\( \beta_2 := -\frac{\mu^2 + (-2 \alpha - 6) \mu + 2 \alpha + 1}{(\mu - 2 \alpha - 5)(\mu - 2 \alpha - 7)} \)

\( \gamma_3 := -\frac{4 \mu - 2 \alpha - 5}{(\mu - 2 \alpha - 9)^2} \)

\( \beta_3 := -\frac{\mu^2 + (-10 - 2 \alpha) \mu - 2 \alpha - 1}{(\mu - 2 \alpha - 7)(\mu - 2 \alpha - 9)} \)

\( \gamma_4 := -\frac{\mu^2 + (10 - 2 \alpha) \mu + 2 \alpha + 1}{(\mu - 2 \alpha - 9)(\mu - 2 \alpha - 11)} \)

\( \gamma_2 n := 2 n^2 (2 n + 2 \alpha + 1 - \mu) / (4 n + 2 \alpha + 1 - \mu)^2 \)
\[
\gamma_{2np1} := \frac{2(n + \alpha + 1)(2n + 1 - \mu)}{(4n + 2\alpha + 3 - \mu)^2}
\]

\[
\beta_{2n} := -\frac{\mu(\mu - 4n - 2\alpha - 2) + 2\alpha + 1}{(4n + 2\alpha + 1 - \mu)(4n + 2\alpha + 3 - \mu)}
\]

\[
\gamma_{2np2} := \frac{2}{(-4n - 2\alpha - 5 + \mu)^2}
\]

\[
\beta_{2np2} := -\frac{\mu^2 - 2\mu\alpha - 4\mu n - 6\mu + 1 + 2\alpha}{(-4n - 2\alpha - 5 + \mu)(\mu - 2\alpha - 7 - 4n)}
\]

\[
\beta_{2np1} := \frac{\mu(\mu - 4n - 2\alpha - 4) - 2\alpha - 1}{(4n + 2\alpha + 3 - \mu)(4n + 2\alpha + 5 - \mu)}
\]

\[
\beta_{2np1} := \frac{-\mu(\mu - 4n - 2\alpha - 2) + 2\alpha + 1}{(4n + 2\alpha + 1 - \mu)(4n + 2\alpha + 3 - \mu)}
\]

\[
E_{2n} := -(2n + 1 - \mu)/(4n + 2\alpha + 3 - \mu)
\]

\[
E_{2n1} := -(2n + 2)/(4n + 2\alpha + 5 - \mu)
\]

\[
E_{2n} := -\frac{2n + 1 - \mu}{4n + 2\alpha + 3 - \mu}
\]

\[
E_{2n1} := -(2n + 2)/(4n + 2\alpha + 5 - \mu)
\]
AN EXAMPLE OF NONSYMMETRIC SEMI-CLASSICAL FORM ...

\[ \gamma_{2np3} := \frac{-2(n+2+\alpha)(\mu-2n-3)}{\mu-2\alpha-7-4n} \]

\[ \beta_{2np3} := \frac{\mu-4\mu n-8\mu-2\mu \alpha - 2\alpha - 1}{(\mu-2\alpha-7-4n)(\mu-2\alpha-9-4n)} \]

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Mohamed Jalel Atia: Faculté des Sciences de Gabès, 6029 Route de Mednine Gabès, Tunisia
E-mail address: jalel.atia@fsg.rnu.tn