PERIODIC SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. The periodic boundary value problems of a class of nonlinear differential equations are investigated.

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1. Introduction. Let us consider the following nonlinear differential equation

$$(\bar{a}(t)\bar{\varphi}_p(x'(t)))' + f(t, x(t)) = 0, \quad (1.1)$$

where $' = d/dt$, $f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous, 2π -periodic in t and $f(t, \cdot) \in \mathbb{C}^1(\mathbb{R}^n, \mathbb{R}^n)$, $\bar{a}(t)\bar{\varphi}_p(x) =: \text{col}(a_1(t)\varphi_p(x_1), \dots, a_n(t)\varphi_p(x_n))$, $a_k(t)$ is 2π -periodic and $a_k(t) \in \mathbb{C}^1(\mathbb{R}, (0, \infty))$, $\varphi_p : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $\varphi_p(s) = |s|^{p-2}s$, with $p > 1$ fixed, $f(t, x) = \text{col}(f_1(t, x), \dots, f_n(t, x))$.

When $p = 2$, $a_k(t) \equiv 1$, $k = 1, 2, \dots, n$. Equation (1.1) is of the form

$$x''(t) + f(t, x) = 0. \quad (1.2)$$

Amaral and Pera [1] and recently Li [5] proved the existence and uniqueness results of (1.2) under the following assumptions:

(L) There exist two constant symmetric $n \times n$ matrices A_0 and B_0 with eigenvalues N_k^2 and $(N_k + 1)^2$, ($k = 1, 2, \dots, n$), respectively, $N_k \geq 0$ is an integer for each k , $D_2 f(t, x)$ is symmetric and there exist two time-dependent continuous symmetric $n \times n$ matrices $A(t)$ and $B(t)$ such that $A_0 \leq A(t) \leq D_2 f(t, x) \leq B(t) \leq B_0$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$. Furthermore, $N_k^2 < \lambda_k(t) \leq \mu_k(t) < (N_k + 1)^2$ on a subset of $[0, 2\pi]$ of positive measure, where $\lambda_k(t)$ and $\mu_k(t)$ are the eigenvalues of $A(t)$ and $B(t)$, respectively.

Inspired by the work of Li [5], we give sufficient conditions for the existence and uniqueness of the 2π -periodic solution of (1.1) by using the initial value problem method and homeomorphism of $\mathbb{R}^n$ to $\mathbb{R}^n$.

2. Initial value problem and eigenvalues problem. Throughout this paper, we denote the interval $[0, 2\pi]$ and M_n denotes the set of all complex $n \times n$ matrices. We also assume the solutions of (1.1) exist on I for any initial value $(x(0), x'(0)) \in \mathbb{R}^{2n}$.

Let us consider the initial value problem

$$u'(t) = g(t, u(t)), \quad u(0) = u_0. \quad (2.1)$$

LEMMA 2.1 [3, 4]. Assume that $g \in \mathbb{C}(I \times \mathbb{R}^n, \mathbb{R}^n)$ and possesses continuous partial derivatives $\partial g / \partial u$ on $I \times \mathbb{R}^n$. Let the solution $u_0(t) = u(t, t_0, u_0)$ of (2.1) exist for $t \in I$ and let

$$H(t, t_0, u_0) = \frac{\partial g(t, u(t, t_0, u_0))}{\partial u}. \quad (2.2)$$

Then

$$\phi(t, t_0, u_0) = \frac{\partial u(t, t_0, u_0)}{\partial u_0} \quad (2.3)$$

exists and is the solution of

$$V' = H(t, t_0, u_0)V \quad (2.4)$$

such that $\phi(t_0, t_0, u_0)$ is the unit matrix.

LEMMA 2.2 [2]. Suppose $A \in M_n$. Then λ is an eigenvalue of the matrix A if and only if $\exp \lambda$ is an eigenvalue of the matrix $\exp A$.

LEMMA 2.3 [2]. If $A \in M_n$ and there exists $\delta > 0$ such that $|\lambda| > \delta$ for all eigenvalues λ of A , then $\|A^{-1}\| \leq \delta^{-n} \|A\|^{n-1}$, where $\|A\| = \max \lambda^{1/2}(A^*A)$ [A^* denotes the adjoint of A , i.e., if $A = (a_{ij})$, $A^* = (\bar{a}_{ji})$].

LEMMA 2.4 [7]. If $A \geq B \geq 0$, and A and B are two real symmetric $n \times n$ matrices, where $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ and $\mu_1 \leq \mu_2 \leq \dots \leq \mu_n$ are eigenvalues of A and B , respectively, then $\lambda_k \geq \mu_k$, for $k = 1, 2, \dots, n$.

LEMMA 2.5 [6]. Assume that $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuously differentiable on $\mathbb{R}^n$ and $\|[F'(x)]^{-1}\| \leq M < +\infty$ for all $x \in \mathbb{R}^n$. Then F is a homeomorphism of $\mathbb{R}^n$ onto $\mathbb{R}^n$.

LEMMA 2.6. Assume A, B are $n \times n$ matrices, then the eigenvalues of the $2n \times 2n$ matrix

$$\begin{pmatrix} 0 & \bar{A} \\ -\bar{B} & 0 \end{pmatrix} \quad (2.5)$$

are the roots of $\det(\lambda^2 I_n + \bar{B}A) = 0$.

PROOF. From the following matrices equality

$$\begin{pmatrix} \lambda I_n & 0 \\ -\bar{B} & \lambda I_n \end{pmatrix} \begin{pmatrix} \lambda I_n & -\bar{A} \\ \bar{B} & \lambda I_n \end{pmatrix} = \begin{pmatrix} \lambda^2 I_n & -\lambda \bar{A} \\ 0 & \lambda^2 I_n + \bar{B}A \end{pmatrix} \quad (2.6)$$

□

we obtain the result of Lemma 2.6 immediately.

3. Main results. Rewrite (1.1) as follows

$$x' = \varphi_q(b(t)y), \quad y' = -f(t, x), \quad (3.1)$$

where $b(t)y =: \text{col}(b_1(t)y_1(t), \dots, b_n(t)y_n(t))$, $b_k(t) = a_k^{-1}(t)$, and $(1/p) + (1/q) = 1$, ($q = p/(p-1)$), $y_k(t) = a_k(t)\varphi_p(x'_k)$, hence $x'_k(t) = \varphi_q(b_k(t)y_k(t))$, $k = 1, 2, \dots, n$.

Let $u = \text{col}(x, y) \in \mathbb{R}^{2n}$, $g(t, u) = \text{col}(\varphi_q(b(t)y), -f(t, x)) \in \mathbb{R}^{2n}$, $v = \text{col}(\alpha, \beta) = \text{col}(x(0), y(0)) = \text{col}(x(0), a(0)\varphi_p(x'(0))) \in \mathbb{R}^{2n}$, then (3.1) is of the form

$$u'(t) = g(t, u(t)), \quad u(0) = v. \quad (3.2)$$

Consider the variation equation of (3.2) with respect to u

$$\xi' = \frac{\partial g(t, u)}{\partial u} \xi, \quad (3.3)$$

where

$$\frac{\partial g(t, u)}{\partial u} = \begin{pmatrix} 0 & \frac{\partial \varphi_q(b(t)y)}{\partial y} \\ -\frac{\partial f(t, x)}{\partial x} & 0 \end{pmatrix} =: \begin{pmatrix} 0 & A \\ -B & 0 \end{pmatrix} \quad (3.4)$$

with

$$A = (q-1) \text{diag}(b_1(t)|y_1(t)|^{q-2}, \dots, b_n(t)|y_n(t)|^{q-2}), \quad (3.5)$$

$$B = \frac{\partial f(t, x(t))}{\partial x} = \nabla f(t, x(t)).$$

Let

$$Z(t) = \exp \int_0^t \frac{\partial g(s, u(s, v))}{\partial u} ds. \quad (3.6)$$

Then $Z(t)$ is a fundamental solution matrix of (3.3) and $Z(0) = I_{2n}$. Meanwhile, by Lemma 2.1 we know that

$$\frac{\partial u}{\partial v} = \begin{pmatrix} \frac{\partial x(t, v)}{\partial \alpha} & \frac{\partial x(t, v)}{\partial \beta} \\ \frac{\partial y(t, v)}{\partial \alpha} & \frac{\partial y(t, v)}{\partial \beta} \end{pmatrix} \quad (3.7)$$

is also a fundamental solution matrix of (3.3). Therefore

$$Z(t) = \frac{\partial u(t, v)}{\partial v}, \quad t \in [0, 2\pi]. \quad (3.8)$$

Define: $h, H: \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$, $h(v) = \text{col}(x(2\pi, v), y(2\pi, v))$,

$$H(v) = v - h(v). \quad (3.9)$$

By Lemma 2.1, $h(v)$ is $\mathbb{C}^1$ -differentiable and so is $H(v)$. Therefore, solving periodic solution of (1.1) is equivalent to finding the fixed points of $h(v)$ or the zero points of $H(v)$. From (3.8) and (3.9)

$$\begin{aligned} H'(v) &= I_{2n} - h'(v) = I_{2n} - \frac{\partial u(2\pi, v)}{\partial v} = I_{2n} - Z(2\pi) \\ &= I_{2n} - \exp \int_0^{2\pi} \frac{\partial g(t, u(t, v))}{\partial u} dt. \end{aligned} \quad (3.10)$$

THEOREM 3.1. *Let $\bar{A}_v = (q-1) \text{diag}(\int_0^{2\pi} b_1(t) |\beta_1 - \int_0^t f_1(\tau, x(\tau)) d\tau|^{q-2} dt, \dots, \int_0^{2\pi} b_n(t) |\beta_n - \int_0^t f_n(x, x(\tau)) d\tau|^{q-2} dt)$, $\bar{B}_v = \int_0^{2\pi} \nabla f(t, x(t)) dt$, where $x(t)$ is any solution of (1.1) satisfying initial conditions $(x(0), y(0)) = v = (\alpha, \beta) \in \mathbb{R}^{2n}$. If there exist $v \in \mathbb{R}^{2n}$ and integers $N_k \geq 0$, $k = 1, 2, \dots, n$, such that the matrix $\bar{B}_v \bar{A}_v$ is similar to a diagonal matrix $C_v = \text{diag}(\lambda_1, \dots, \lambda_n)$ with $(2\pi N_k)^2 < \lambda_k < [2\pi(N_k + 1)]^2$, $k = 1, 2, \dots, n$. Then (1.1) has a unique 2π -periodic solution $x(t)$ satisfying the initial condition $(x(0), y(0)) = v$.*

PROOF. By Lemma 2.5, we need only to show that $H'(V)$ is invertible and that there exists a constant $M > 0$ such that $\|[H'(v)]^{-1}\| \leq M$.

In fact, from (3.1) and (3.5), $A = (q-1) \text{diag}(b_1(t) |y_1(t)|^{q-2}, \dots, b_n(t) |y_n(t)|^{q-2})$, since

$$y_k(t) = y_k(0) - \int_0^t f_k(\tau, x(\tau)) d\tau = \beta_k - \int_0^t f_k(\tau, x(\tau)) d\tau, \quad k = 1, 2, \dots, n \quad (3.11)$$

we have

$$\begin{aligned} \bar{A}_v = \int_0^{2\pi} A dt = (q-1) \text{diag} \left(\int_0^{2\pi} b_1(t) \left| \beta_1 - \int_0^t f_1(\tau, x(\tau)) d\tau \right|^{q-2} dt, \dots, \right. \\ \left. \times \int_0^{2\pi} b_n(t) \left| \beta_n - \int_0^t f_n(\tau, x(\tau)) d\tau \right|^{q-2} dt \right). \end{aligned} \quad (3.12)$$

From Lemma 2.6, the eigenvalues of the matrix $\begin{pmatrix} 0 & \bar{A}_v \\ -\bar{B}_v & 0 \end{pmatrix}$ are $\pm\sqrt{\lambda_1}i, \pm\sqrt{\lambda_2}i, \dots, \pm\sqrt{\lambda_n}i$. By (3.5), (3.10), and Lemma 2.2 the eigenvalues of $H'(v)$ are

$$\mu_k = 1 - \exp(\pm\sqrt{\lambda_k}i) = 1 - \cos\sqrt{\lambda_k} \mp i \sin\sqrt{\lambda_k}, \quad k = 1, 2, \dots, n. \quad (3.13)$$

From the assumption of λ_k and

$$|\mu_k| = \sqrt{2 - 2\cos\sqrt{\lambda_k}} = 2 \left| \sin \frac{\sqrt{\lambda_k}}{2} \right| \quad (3.14)$$

it follows that

$$|\mu_k| \geq 2 \min_{1 \leq k \leq n} \left(\left| \sin \frac{\sqrt{\lambda_k}}{2} \right| \right) > 0 \quad (3.15)$$

because $N_k\pi < \sqrt{\lambda_k}/2 < (N_k+1)\pi$, $k = 1, 2, \dots, n$,

$$\|[H'(v)]^{-1}\| \leq \frac{1 + \exp(4\max_{1 \leq k \leq n}(N_k+1)^2\pi^2)}{(2 \min \{ |\sin \sqrt{\lambda_k}/2| \})^n} = M. \quad (3.16)$$

Now, from Lemma 2.5, $H'(v)$ is invertible, since by Lemma 2.3, $H'(v)$ is homeomorphism of $\mathbb{R}^n$ onto $\mathbb{R}^n$, hence there exists a unique $v_0 \in \mathbb{R}^n$ such that $H(v_0) = 0$, that is, $h(v_0) = v_0$. Theorem 3.1 is proved. $\square$

COROLLARY 3.2. *Let $p = 2$, $a_k(t) \equiv 1$, $k = 1, 2, \dots, n$ in (1.1), and suppose (L) holds, then (1.1) has a unique 2π -periodic solution.*

PROOF. In this case, $q = 2$, hence $\tilde{A} = 2\pi I_n$, then eigenvalues of $\begin{pmatrix} 0 & \tilde{A} \\ -\tilde{B} & 0 \end{pmatrix}$ are

$$\pm\sqrt{2\pi\lambda_k}i, \quad k = 1, 2, \dots, n \quad (3.17)$$

with

$$2\pi N_k^2 < \lambda_k < 2\pi(N_k + 1)^2, \quad (3.18)$$

therefore

$$\| [H'(v)]^{-1} \| \leq \frac{1 + \exp(2\max_{1 \leq k \leq N} (N_k + 1)^2 \pi)}{(2\min_{1 \leq k \leq N} \{ \sin |\sqrt{2\pi a_k}/2|, \sin |\sqrt{2\pi b_k}/2| \})^2} = M, \quad (3.19)$$

where

$$2\pi N_k^2 < a_k = \int_0^{2\pi} \lambda_k(t) dt \leq \int_0^{2\pi} \mu_k(t) dt = b_k < 2\pi(N_k + 1)^2. \quad (3.20)$$

From Lemma 2.3 again, (1.1) has a unique 2π -periodic solution. Corollary 3.2 is proved. $\square$

REMARK 3.3. Corollary 3.2 is the result of [1] and [5].

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