LOCAL PROPERTIES OF FOURIER SERIES

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ABSTRACT. A theorem on local property of $|\tilde{N}, p_n|_k$ summability of factored Fourier series, which generalizes some known results, and also a general theorem concerning the $|\tilde{N}, p_n|_k$ summability factors of Fourier series have been proved.

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1. Introduction. Let $\sum a_n$ be a given infinite series with partial sums (s_n) . Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^n p_v \rightarrow \infty \quad \text{as } n \rightarrow \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1). \quad (1.1)$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{v=0}^n p_v s_v \quad (1.2)$$

defines the sequence (t_n) of the $(\tilde{N}, p_n)$ mean of the sequence (s_n) generated by the sequence of coefficients (p_n) (see [8]).

The series $\sum a_n$ is said to be summable $|\tilde{N}, p_n|_k, k \geq 1$, if (see [4])

$$\sum_{n=1}^{\infty} \left(\frac{p_n}{P_n} \right)^{k-1} |t_n - t_{n-1}|^k < \infty. \quad (1.3)$$

In the special case when $p_n = 1$ for all values of n (resp., $k = 1$), $|\tilde{N}, p_n|_k$ summability is the same as $|C, 1|$ (resp., $|\tilde{N}, p_n|$) summability. Also if we take $k = 1$ and $p_n = 1/n$ summability $|\tilde{N}, p_n|_k$ is equivalent to the summability $|R, \log n, 1|$. A sequence (λ_n) is said to be convex if $\Delta^2 \lambda_n \geq 0$ for every positive integer n , where $\Delta^2 \lambda_n = \Delta \lambda_n - \Delta \lambda_{n+1}$ and $\Delta \lambda_n = \lambda_n - \lambda_{n+1}$.

Let $f(t)$ be a periodic function with period 2π , and integrable (L) over $(-\pi, \pi)$. Without loss of generality we may assume that the constant term in the Fourier series of $f(t)$ is zero, so that

$$\int_{-\pi}^{\pi} f(t) dt = 0 \quad (1.4)$$

and

$$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t) \quad (1.5)$$

It is familiar that the convergence of the Fourier series at $t = x$ is a local property of f (i.e., it depends only on the behaviour of f in an arbitrarily small neighbourhood of x), and hence the summability of the Fourier series at $t = x$ by any regular linear summability method is also a local property of f . The local property problem of the factored Fourier series have been studied by several authors (see [1, 2, 5, 6, 7, 9]). Few of them are given below.

2. Mohanty [13] has demonstrated that the $|R, \log n, 1|$ summability of the factored Fourier series

$$\sum \frac{A_n(t)}{\log(n+1)} \quad (2.1)$$

at $t = x$, is a local property of the generating function of f , whereas the $|C, 1|$ summability of this series is not. Later on, Matsumoto [10] improved this result by replacing the series (2.1) by

$$\sum \frac{A_n(t)}{[\log \log(n+1)]^\delta}, \quad \delta > 1. \quad (2.2)$$

Generalizing the above result Bhatt [3] proved the following theorem.

THEOREM 2.1. *If (λ_n) is a convex sequence such that $\sum n^{-1} \lambda_n$ is convergent, then the summability $|R, \log n, 1|$ of the series $\sum A_n(t) \lambda_n \log n$ at a point can be ensured by a local property.*

Mishra [12] has proved the following theorem by replacing the factor $(\lambda_n \log n)$ in the most general form.

THEOREM 2.2. *Let the sequence (p_n) be such that*

$$P_n = O(np_n), \quad P_n \Delta p_n = O(p_n p_{n+1}). \quad (2.3)$$

Then the summability $|\tilde{N}, p_n|$ of the series

$$\sum_{n=1}^{\infty} \frac{A_n(t) \lambda_n P_n}{np_n}, \quad (2.4)$$

at a point can be ensured by local property, where (λ_n) is as in Theorem 2.1.

But this theorem does not directly generalize any of the above mentioned results involving $|R, \log n, 1|$ summability since order relations are not satisfied by $p_n = 1/n$.

3. The aim of this paper is to prove a more general theorem which includes some of the above mentioned results as special cases.

Now, we shall prove the following theorem.

THEOREM 3.1. *Let $k \geq 1$. If (λ_n) is a convex sequence such that $\sum p_n \lambda_n$ is convergent, then the summability $|\tilde{N}, p_n|_k$ of the series $\sum A_n(t) \lambda_n P_n$ at a point can be ensured by a local property.*

We need the following lemmas for the proof of our theorem.

LEMMA 3.2 [11]. *If (λ_n) is a convex sequence such that $\sum p_n \lambda_n$ is convergent, where (p_n) is a sequence of positive numbers such that $P_n \rightarrow \infty$ as $n \rightarrow \infty$, then (λ_n) is a non-negative monotonic decreasing sequence tending to zero, $P_n \lambda_n = o(1)$ as $n \rightarrow \infty$ and $\sum P_n \Delta \lambda_n < \infty$.*

LEMMA 3.3. *Let $k \geq 1$. If (λ_n) is a convex sequence such that $\sum p_n \lambda_n$ is convergent and (s_n) is bounded, then the series $\sum a_n \lambda_n P_n$ is summable $|\bar{N}, p_n|_k$.*

PROOF. Let (T_n) be the sequence of $(\bar{N}, p_n)$ mean of the series $\sum a_n \lambda_n P_n$. Then, by definition, we have

$$T_n = \frac{1}{P_n} \sum_{v=0}^n p_v \sum_{r=0}^v a_r \lambda_r P_r = \frac{1}{P_n} \sum_{v=0}^n (P_n - P_{v-1}) a_v \lambda_v P_v. \quad (3.1)$$

Then, for $n \geq 1$, we have

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^n P_{v-1} P_v a_v \lambda_v. \quad (3.2)$$

By Abel's transformation, we have

$$\begin{aligned} T_n - T_{n-1} &= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v P_v s_v \Delta \lambda_v - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v s_v p_v \lambda_v \\ &\quad - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v p_{v+1} s_v \lambda_{v+1} + s_n p_n \lambda_n \\ &= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}. \end{aligned} \quad (3.3)$$

By Minkowski's inequality for $k > 1$, to complete the proof of Lemma 3.3, it is sufficient to show that

$$\sum_{n=1}^{\infty} \left(\frac{p_n}{P_n} \right)^{k-1} |T_{n,r}|^k < \infty, \quad \text{for } r = 1, 2, 3, 4. \quad (3.4)$$

Now, applying Hölder's inequality with indices k and k' , where $1/k + 1/k' = 1$, we get that

$$\sum_{n=2}^{m+1} \left(\frac{p_n}{P_n} \right)^{k-1} |T_{n,1}|^k \leq \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \left\{ \sum_{v=1}^{n-1} |s_v|^k P_v P_v \Delta \lambda_v \right\} \times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_v P_v \Delta \lambda_v \right\}^{k-1}. \quad (3.5)$$

Since

$$\sum_{v=1}^{n-1} P_v P_v \Delta \lambda_v \leq P_{n-1} \sum_{v=1}^{n-1} P_v \Delta \lambda_v, \quad (3.6)$$

it follows by Lemma 3.2 that

$$\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_v P_v \Delta \lambda_v \leq \sum_{v=1}^{n-1} P_v \Delta \lambda_v = O(1) \quad \text{as } m \rightarrow \infty. \quad (3.7)$$

Therefore

$$\begin{aligned}
 \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{k-1} |T_{n,1}|^k &= O(1) \sum_{n=2}^{m+1} \frac{P_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} |s_v|^k P_v P_v \Delta \lambda_v \\
 &= O(1) \sum_{v=1}^m |s_v|^k P_v P_v \Delta \lambda_v \sum_{n=v+1}^{m+1} \frac{P_n}{P_n P_{n-1}} \\
 &= O(1) \sum_{v=1}^m P_v \Delta \lambda_v = O(1) \quad \text{as } m \rightarrow \infty,
 \end{aligned} \tag{3.8}$$

by virtue of the hypotheses and Lemma 3.2. Again

$$\begin{aligned}
 \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{k-1} |T_{n,2}|^k &\leq \sum_{n=2}^{m+1} \frac{P_n}{P_n P_{n-1}} \left\{ \sum_{v=1}^{n-1} |s_v|^k (P_v \lambda_v)^k p_v \right\} \times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1} \\
 &= O(1) \sum_{v=2}^{m+1} \frac{P_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} |s_v|^k (P_v \lambda_v)^k p_v \\
 &= O(1) \sum_{v=1}^m |s_v|^k (P_v \lambda_v)^k p_v \sum_{n=v+1}^{m+1} \frac{P_n}{P_n P_{n-1}} \\
 &= O(1) \sum_{v=1}^m |s_v|^k (P_v \lambda_v)^k \frac{p_v}{P_v} \\
 &= O(1) \sum_{v=1}^m |s_v|^k (P_v \lambda_v)^{k-1} p_v \lambda_v \\
 &= O(1) \sum_{v=1}^m p_v \lambda_v = O(1) \quad \text{as } m \rightarrow \infty,
 \end{aligned} \tag{3.9}$$

by virtue of the hypotheses and Lemma 3.2. Using the fact that $P_v < P_{v+1}$, similarly we have that

$$\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{k-1} |T_{n,3}|^k = O(1) \sum_{v=1}^m p_{v+1} \lambda_{v+1} = O(1) \quad \text{as } m \rightarrow \infty, \tag{3.10}$$

Finally, we have that

$$\begin{aligned}
 \sum_{n=1}^m \left(\frac{P_n}{p_n} \right)^{k-1} |T_{n,4}|^k &= \sum_{n=1}^m |s_n|^k (P_n \lambda_n)^{k-1} p_n \lambda_n \\
 &= O(1) \sum_{n=1}^m p_n \lambda_n = O(1) \quad \text{as } m \rightarrow \infty,
 \end{aligned} \tag{3.11}$$

by virtue of the hypotheses and Lemma 3.2. Therefore, we get that

$$\sum_{n=1}^m \left(\frac{P_n}{p_n} \right)^{k-1} |T_{n,r}|^k = O(1) \quad \text{as } m \rightarrow \infty, \text{ for } r = 1, 2, 3, 4. \tag{3.12}$$

This completes the proof of Lemma 3.3. □

In the particular case if we take $p_n = 1$ for all values of n in this lemma, then we get the following corollary.

COROLLARY 3.4. *Let $k \geq 1$. If (λ_n) is a convex sequence such that $\sum \lambda_n$ is convergent and (s_n) is bounded, then the series $\sum na_n \lambda_n$ is summable $|C, 1|_k$.*

PROOF OF THEOREM 3.1. Since the behaviour of the Fourier series, as far as convergence is concerned, for a particular value of x depends on the behaviour of the function in the immediate neighbourhood of this point only, hence the truth of the theorem is a necessary consequence of Lemma 3.3. If we take $p_n = 1$ for all values of n in this theorem, then we get a local property result concerning the $|C, 1|_k$ summability. $\square$

Now we shall prove the following theorem for $|\bar{N}, p_n|_k$ summability factors of Fourier series.

THEOREM 3.5. *Let $k \geq 1$ and let (λ_n) be a convex sequence such that $\sum p_n \lambda_n < \infty$, where (p_n) is a sequence of positive numbers such that $P_n \rightarrow \infty$. If $\sum_{v=1}^n P_v A_v(t) = O(P_n)$, then the series $\sum A_n(t) P_n \lambda_n$ is summable $|\bar{N}, p_n|_k$, where $A_v(t)$ is as in (1.5).*

PROOF. Let $T_n(t)$ denotes the $(\bar{N}, p_n)$ mean of the series $\sum A_n(t) P_n \lambda_n$. Then, by definition, we have

$$T_n = \frac{1}{P_n} \sum_{v=0}^n p_v \sum_{r=0}^v A_r(t) P_r \lambda_r = \frac{1}{P_n} \sum_{v=0}^n (P_n - P_{v-1}) A_v(t) \lambda_v P_v. \quad (3.13)$$

Then, for $n \geq 1$, we have

$$T_n(t) - T_{n-1}(t) = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^n P_{v-1} P_v A_v(t) \lambda_v. \quad (3.14)$$

By Abel's transformation, we have

$$\begin{aligned} T_n(t) - T_{n-1}(t) &= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \Delta(P_{v-1} \lambda_v) \sum_{r=1}^v P_r A_r(t) + \frac{p_n}{P_n} \lambda_n \sum_{v=1}^n P_v A_v(t) \\ &= O(1) \left\{ \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} (P_v \lambda_v - p_v \lambda_v - P_v \lambda_{v+1}) P_v \right\} + O(1) p_n \lambda_n \\ &= O(1) \left\{ \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v P_v \lambda_v - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v p_v \lambda_v + p_n \lambda_n \right\} \\ &= O(1) \{T_{n,1}(t) + T_{n,2}(t) + T_{n,3}(t)\}. \end{aligned} \quad (3.15)$$

Since

$$|T_{n,1}(t) + T_{n,2}(t) + T_{n,3}(t)|^k \leq 3^k \{|T_{n,1}(t)|^k + |T_{n,2}(t)|^k + |T_{n,3}(t)|^k\}, \quad (3.16)$$

to complete the proof of Theorem 3.5, it is sufficient to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} |T_{n,r}(t)|^k < \infty, \quad \text{for } r = 1, 2, 3. \quad (3.17)$$

Now, applying Hölder's inequality with indices k and k' , where $1/k + 1/k' = 1$ and by using (3.7), we get that

$$\begin{aligned}
 \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{k-1} |T_{n,1}(t)|^k &\leq \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \left\{ \sum_{v=1}^{n-1} P_v P_v \Delta \lambda_v \right\} \times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_v P_v \Delta \lambda_v \right\}^{k-1} \\
 &= O(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v P_v \Delta \lambda_v \\
 &= O(1) \sum_{v=1}^m P_v P_v \Delta \lambda_v \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}} \\
 &= O(1) \sum_{v=1}^m P_v \Delta \lambda_v = O(1) \quad \text{as } m \rightarrow \infty,
 \end{aligned} \tag{3.18}$$

$$\begin{aligned}
 \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} \right)^{k-1} |T_{n,2}(t)|^k &\leq \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \left\{ \sum_{v=1}^{n-1} (P_v \lambda_v)^k p_v \right\} \times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1} \\
 &= O(1) \sum_{v=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} (P_v \lambda_v)^k p_v \\
 &= O(1) \sum_{v=1}^m (P_v \lambda_v)^k p_v \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}} \\
 &= O(1) \sum_{v=1}^m (P_v \lambda_v)^k \frac{p_v}{P_v} \\
 &= O(1) \sum_{v=1}^m (P_v \lambda_v)^{k-1} p_v \lambda_v \\
 &= O(1) \sum_{v=1}^m p_v \lambda_v = O(1) \quad \text{as } m \rightarrow \infty,
 \end{aligned} \tag{3.19}$$

by virtue of the hypotheses and Lemma 3.2. Finally, as in $T_{n,1}(t)$, we have that

$$\begin{aligned}
 \sum_{n=1}^m \left(\frac{P_n}{p_n} \right)^{k-1} |T_{n,3}(t)|^k &= \sum_{n=1}^m (P_n \lambda_n)^{k-1} p_n \lambda_n \\
 &= O(1) \sum_{n=1}^m p_n \lambda_n = O(1) \quad \text{as } m \rightarrow \infty,
 \end{aligned} \tag{3.20}$$

Therefore, we get that

$$\sum_{n=1}^m \left(\frac{P_n}{p_n} \right)^{k-1} |T_{n,r}(t)|^k = O(1) \quad \text{as } m \rightarrow \infty, \text{ for } r = 1, 2, 3. \tag{3.21}$$

This completes the proof of Theorem 3.5. $\square$

As a special case the following results can be obtained from Theorem 3.5.

(1) If we take $p_n = 1$ for all values of n , then we get a result concerning the $|C, 1|_k$ summability factors of Fourier series.

(2) If we take $k = 1$ and $p_n = 1/n$, then we get another new result related to $|R, \log n, 1|$ summability factors of Fourier series.

REFERENCES

- [1] S. Baron, *Local property of absolute summability of Fourier series*, Tartu Riikl. Ül. Toime-
tised Vih. **177** (1965), 106–120. MR 35#2073. Zbl 166.07201.
- [2] S. N. Bhatt, *An aspect of local property of $|R, \log n, 1|$ summability of Fourier series*, To-
hoku Math. J. (2) **11** (1959), 13–19. MR 21#6504. Zbl 107.05303.
- [3] ———, *An aspect of local property of $|R, \log n, 1|$ summability of the factored Fourier se-
ries*, Proc. Nat. Inst. Sci. India Part A **26** (1960), 69–73. MR 28#1444. Zbl 098.04401.
- [4] H. Bor, *On two summability methods*, Math. Proc. Cambridge Philos. Soc. **97** (1985), no. 1,
147–149. MR 86d:40004. Zbl 554.40008.
- [5] ———, *Local property of $|\tilde{N}, p_n|_k$ summability of factored Fourier series*, Bull. Inst. Math.
Acad. Sinica **17** (1989), no. 2, 165–170. MR 91e:42007. Zbl 697.42006.
- [6] ———, *On the local property of $|\tilde{N}, p_n|_k$ summability of factored Fourier series*, J. Math.
Anal. Appl. **163** (1992), no. 1, 220–226. MR 93b:42013. Zbl 764.42002.
- [7] D. Borwein, *The nonlocal nature of the summability of Fourier series by certain absolute
Riesz methods*, Proc. Amer. Math. Soc. **114** (1992), no. 1, 89–94. MR 92c:42006.
Zbl 753.42006.
- [8] G. H. Hardy, *Divergent Series*, Oxford, at the Clarendon Press, 1949. MR 11,25a.
Zbl 032.05801.
- [9] M. Izumi and S. Izumi, *Localization problem of the absolute Riesz and absolute Nörlund
summabilities of Fourier series*, Canad. J. Math. **22** (1970), 615–625. MR 41#8915.
Zbl 205.08501.
- [10] K. Matsumoto, *Local property of the summability $|R, \lambda_n, 1|$* , Tohoku Math. J. (2) **8** (1956),
114–124. MR 18,208b. Zbl 071.28403.
- [11] S. M. Mazhar, *On the summability factors of infinite series*, Publ. Math. Debrecen **13** (1966),
229–236. MR 35#3313. Zbl 166.31602.
- [12] K. N. Mishra, *Multipliers for $|\tilde{N}, p_n|$ summability of Fourier series*, Bull. Inst. Math. Acad.
Sinica **14** (1986), no. 4, 431–438. MR 88e:42013. Zbl 635.42011.
- [13] R. Mohanty, *On the summability $|R, \log w, 1|$ of a Fourier series*, J. London Math. Soc. **25**
(1950), 67–72. MR 11,592b. Zbl 035.33602.

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