ON THE DEGREE OF APPROXIMATION BY GAUSS WEIERSTRASS INTEGRALS

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(Received 1 July 1998 and in revised form 15 October 1998)

Abstract. We obtain the degree of approximation of functions belonging to class \( \text{Lip}(\psi(u,v);p) \), \( p > 1 \) using the Gauss Weierstrass integral of the double Fourier series of \( f(x,y) \).

Keywords and phrases. Lip(\( \psi(u,v);p \) class, Gauss Weierstrass integral, arithmetic means, Jackson operator, degree of approximation, Fourier series, Hölder’s inequality.

2000 Mathematics Subject Classification. Primary 41A25.

1. Introduction and results. Let the function \( f(x,y) \) be integrable in the sense of Lebesgue over the square \((-H^5115,\pi;0,2\pi)\) and periodic with period \(2\pi\) in each variable outside the square. Let the double Fourier series of \( f(x,y) \) be

\[
\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{m,n}(x,y),
\]

where

\[
\begin{align*}
A_{0,0}(x,y) &= \frac{1}{4} a_{0,0}, \\
A_{m,0}(x,y) &= \frac{1}{2} (a_{m,0} \cos mx + b_{m,0} \sin mx), \\
A_{0,n}(x,y) &= \frac{1}{2} (a_{0,n} \cos nx + b_{0,n} \sin nx), \\
A_{m,n}(x,y) &= a_{m,n} \cos mx \cos ny + b_{m,n} \cos mx \sin ny + c_{m,n} \sin mx \cos ny + d_{m,n} \sin mx \sin ny.
\end{align*}
\]

Let

\[
\phi(u,v) = \frac{1}{4} \left\{ f(x+u,y+v) + f(x-u,y+v) + f(x+u,y-v) + f(x-u,y-v) - 4f(x,y) \right\},
\]

The series

\[
\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \lambda_{m,n} A_{m,n}(x,y)
\]

is called the double Fourier series associated with the function \( \phi(u,v) \) such that
\[ \lambda_{m,n} = \begin{cases} 
\frac{1}{4} & \text{for } m = 0, n = 0, \\
\frac{1}{2} & \text{for } m = 0, n > 0; m > 0, n = 0, \\
1 & \text{for } m > 0, n > 0. 
\end{cases} \] (1.5)

Also, the coefficients in the series (1.4) are given by

\[ a_{m,n} = \frac{1}{\pi^2} \int_0^{2\pi} \int_0^{2\pi} \phi(u,v) \cos mu \cos nv \, du \, dv \] (1.6)

and three other similar expressions defining \( b_{m,n}, c_{m,n}, d_{m,n} \).

We define the Gauss Weierstrass integral of \( f(x,y) \) by

\[ W_{m,n}(x,y) = W(x,y; \xi,\eta) = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{f(x+t,y+s)}{\psi(t,s)^{1/p}} \, dt \, ds + o(\xi,\eta), \] (1.7)

where \( o(\xi,\eta) \to 0 \) as \( \xi \to 0, \eta \to 0 \), and \( (\xi,\eta) \) is the product of \( \xi \) and \( \eta \).

A \( 2\pi \) periodic function \( f(x,y) \) in each variable \( x \) and \( y \) is said to belong to the class \( \text{Lip}(\psi(u,v);p) \), \( p > 1 \), [2] if

\[ |f(x+u,y+v) - f(x,y)| \leq M \left( \frac{\psi(u,v)}{(u,v)^{1/p}} \right), \quad 0 < u < \pi, 0 < v < \pi, \] (1.8)

where \( \psi(u,v) \) is a positive increasing function of the variables \( u, v \) and \( M \) is a positive number independent of \( x, u \) and \( v \).

Yoshimitsu, [1] proved a theorem for obtaining the degree of approximation of class of functions \( \text{Lip}(\alpha,\beta) \), \( 0 < \alpha < 1 \) and \( 0 < \beta < 1 \), by means of the first arithmetic means of double Fourier series. Siddiqi and Mohammadzadeh [3] extended the result in two directions in terms of a positive increasing function of two variables. Recently Khan, [2] extended the result of Yoshimitsu, Siddiqi et al. for the more general operator, the Jackson type operator, and more general class of functions, \( \text{Lip}(\psi(u,v);p) \), \( p > 1 \). The object of the present paper is to determine the degree of approximation for the functions belonging to the class \( \text{Lip}(\psi(u,v);p) \), \( p > 1 \), by means of Gauss Weierstrass integral of the double Fourier series of \( f(x,y) \).

Our theorem states as follows.

**Theorem 1.1.** Let \( f(x,y) \) be a continuous function of period \( 2\pi \) with respect to each variable \( x \) and \( y \) belonging to \( \text{Lip}(\psi(u,v);p) \), \( p > 1 \) class, then

\[ |W(x,y;\xi,\eta) - f(x,y)| = c\left( \frac{\psi(\xi,\eta)}{(\xi,\eta)^{(1/p) - (1/2)}} \right), \] (1.9)

provided

\[ \left[ \int_0^\xi \int_0^\eta \left( \frac{\psi(t,s)}{(t,s)^{1/p}} \right)^p \, dt \, ds \right]^{1/p} = c(\psi(\xi,\eta)), \] (1.10)
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\[
\left[ \int_0^\pi \int_\pi^\pi (\frac{\psi(t,s)}{(t,s)^{2+(1/p)}})^p \, dt \, ds \right]^{1/p} = \mathcal{O}\left(\frac{\psi(\xi,\eta)}{(\xi,\eta)^{2}}\right).
\] (1.11)

2. Proof of the theorem. Using (1.7), we get

\[
W(x,y;\xi,\eta) - f(x,y)
= \sqrt{\frac{\pi}{(\xi,\eta)}} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \phi(t,s)e^{-(t^2/\xi)+(s^2/\eta)} \, dt \, ds + R(x,y;\xi,\eta)
= \sqrt{\frac{\pi}{(\xi,\eta)}} \int_0^\pi \int_0^\pi \phi(t,s)e^{-(t^2/\xi)+(s^2/\eta)} \, dt \, ds + \mathcal{O}(\xi,\eta)
= \sqrt{\frac{\pi}{(\xi,\eta)}} \left( \int_0^\pi \int_0^\pi \phi(t,s)e^{-(t^2/\xi)+(s^2/\eta)} \, dt \, ds \right)
= I_{1,1}(x,y) + I_{1,2}(x,y) + I_{2,2}(x,y) + I_{1,3}(x,y) + I_{2,3}(x,y).
\] (2.1)

Applying Hölder’s inequality of two variables and the fact that \( \phi(u,v) \in \text{Lip}(\psi(u,v);p), p > 1 \) for \( I_{1,2}(x,y) \), we get

\[
|I_{1,2}(x,y)| \leq 4 \sqrt{\frac{\pi}{(\xi,\eta)}} \left[ \int_0^\pi \int_0^\pi |\phi(t,s)|^p \, dt \, ds \right]^{1/p} \cdot \left[ \int_0^\pi \int_0^\pi (e^{-(t^2/\xi)+(s^2/\eta)})^{p'} \, dt \, ds \right]^{1/p'},
\] (2.2)

where

\[
\frac{1}{p'} = \frac{p-1}{p}.
\]

\[
\leq 4 \sqrt{\frac{\pi}{(\xi,\eta)}} \left[ \int_0^\pi \int_0^\pi \left( \frac{\psi(t,s)}{(t,s)^{2+(1/p)}} \right)^p \, dt \, ds \right]^{1/p} \left[ \int_0^\pi \int_0^\pi (e^{-(t^2/\xi)+(s^2/\eta)})^{p'} \, dt \, ds \right]^{1/p'}
= \mathcal{O}\left(\frac{\psi(\xi,\eta)}{(\xi,\eta)^{2}}\right) \left( \int_0^\pi \int_0^\pi e^{-(t^2/\xi)+(s^2/\eta)} \, dt \, ds \right)^{1/p'} \text{ using condition (1.10)}
= \mathcal{O}\left(\frac{\psi(\xi,\eta)}{(\xi,\eta)^{2}}\right) \left( \int_0^\pi \int_0^\pi e^{-(t^2/\xi)+(s^2/\eta)} \, dt \, ds \right)^{1/p'}
= \mathcal{O}\left(\frac{\psi(\xi,\eta)}{(\xi,\eta)^{2}}\right) \mathcal{O}(\xi,\eta)^{1/p'} = \mathcal{O}\left(\frac{\psi(\xi,\eta)}{(\xi,\eta)^{(1/p)-(1/2)}}\right),
\] (2.3)

where \( u = t\sqrt{p'/\xi} \) and \( v = s\sqrt{p'/\eta} \).

Applying Hölder’s inequality of two variables to \( I_{1,3}(x,y) \) and using the fact that \( \phi(u,v) \in \text{Lip}(\psi(u,v);p), p > 1 \), we get
\[ |I_{1,3}(x,y)| \leq \frac{4\sqrt{\pi}}{\sqrt{\xi, \sqrt{\eta}}} \left[ \left( \int_{\xi}^{\eta} \phi(t,s) \frac{1}{(t,s)^2} \ dt \ ds \right)^{1/p} \left( \int_{\xi}^{\eta} \frac{e^{-(t^2/\xi)+(s^2/\eta)p'}}{(t,s)^{2p'}} \ dt \ ds \right)^{1/p'} \right] \]
\[ \leq \frac{4\sqrt{\pi}}{\sqrt{\xi, \sqrt{\eta}}} \left\{ \left[ \int_{\xi}^{\eta} \left( \frac{\psi(t,s)}{(t,s)^{2+(1/p)}} \right)^p \ dt \ ds \right]^{1/p} \cdot \left( \int_{\xi}^{\eta} \left( \int_0^\pi \left( \left| \phi(t,s) \right| \right)^p e^{-t^2/\xi} \ dt \ ds \right)^{1/p'} \right) \right\} \]
\[ = C \left( \frac{\psi(\xi, \eta)}{(\xi, \eta)^{3/2}} \right)^{1/p} \left[ \left[ \int_{\xi}^{\eta} e^{-u^2} \left( \sqrt{\xi} \right)^{2p} \left( \sqrt{\eta} \right)^{1/p} du \right]^{1/p} \cdot \left[ \int_{\xi}^{\eta} \left( \int_0^\pi \left( \left| \phi(t,s) \right| \right)^p e^{-s^2/\eta} \ ds \right)^{1/p'} \right) \right] \]
\[ = C \left( \frac{\psi(\xi, \eta)}{(\xi, \eta)^{3/2}} \right)^{1/p} \left[ \left( \frac{\psi(\xi, \eta)}{(\xi, \eta)^{1/p-1/2}} \right) \right]. \]

Similarly, we can prove that
\[ |I_{4,2}(x,y)| = C \left( \frac{\psi(\xi, \eta)}{(\xi, \eta)^{3/2}} \right), \quad |I_{4,3}(x,y)| = C \left( \frac{\psi(\xi, \eta)}{(\xi, \eta)^{3/2}} \right). \]

Finally, we get
\[ W(x,y;\xi, \eta) - f(x,y) = C \left( \frac{\psi(\xi, \eta)}{(\xi, \eta)^{3/2}} \right). \]

**Remark 2.1.** It may also be remarked that by giving different values to \( \psi(u,v) \), we get some interesting results:

(i) If \( \psi(u,v) = u^\alpha v^\beta, [1], 0 < \alpha < 1, 0 < \beta < 1 \), then we have
\[ |W(x,y;\xi, \eta) - f(x,y)| = C(\xi^{\alpha+(1/2)-(1/p)} \times \eta^{\beta+(1/2)-(1/p)}). \]

(ii) If \( \psi(u,v) = J(u,v)(u,v)^ {1/p}, [3], where J(u,v) is a positive increasing function of variables u and v, then
\[ |W(x,y;\xi, \eta) - f(x,y)| = C \left( \frac{J(\xi, \eta)}{(\xi, \eta)^{1/2}} \right). \]

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