SUBORDINATION BY CONVEX FUNCTIONS

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ABSTRACT. Let $K(\alpha)$, $0 \leq \alpha < 1$, denote the class of functions $g(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are regular and univalently convex of order α in the unit disc U . Pursuing the problem initiated by Robinson in the present paper, among other things, we prove that if f is regular in U , $f(0) = 0$, and $f(z) + zf'(z) < g(z) + zg'(z)$ in U , then (i) $f(z) < g(z)$ at least in $|z| < r_0$, $r_0 = \sqrt{5}/3 = 0.745\dots$ if $f \in K$; and (ii) $f(z) < g(z)$ at least in $|z| < r_1$, $r_1((51 - 24\sqrt{2})/23)^{1/2} = 0.8612\dots$ if $g \in K(1/2)$.

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1. Introduction. Let S denote the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ that are regular and univalent in the unit disc $U = \{z/|z| < 1\}$. For a given α , $0 \leq \alpha < 1$, denote by $K(\alpha)$ the subclass of S consisting of functions f which satisfy the condition

$$\operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha, \quad z \in U. \quad (1.1)$$

$K(\alpha)$ is called the class of convex functions of order α and $K = K(0)$ is the class of convex functions.

Suppose that f and g are regular in $|z| < \rho$ and $f(0) = g(0)$. In addition, suppose that g is also univalent in $|z| < \rho$. We say that f is subordinate to g in $|z| < \rho$ (in symbols, $f(z) < g(z)$ in $|z| < \rho$) if $f(|z| < \rho) \subset g(|z| < \rho)$.

In 1947, Robinson [2] proved that if $g(z) + zg'(z)$ is in S and $f(z) + zf'(z) < g(z) + zg'(z)$ in $|z| < 1$, then $f(z) < g(z)$ at least in $|z| < r_0 = 1/5$. Subsequently, Singh and Singh [4] increased the constant r_0 to $2 - \sqrt{3} = 0.268\dots$. Miller, Mocanu, and Read [1] further increased the constant to $4 - \sqrt{13} = 0.3944\dots$.

Here, we consider the problem of Robinson when $g \in K$ and $K(1/2)$, respectively. (It is easy to see that $g(z) + zg'(z)$ is close-to-convex and hence univalent in $|z| < 1$ when $g \in K$.) We remark that our method works even when $g \in K(\alpha)$. However, calculations in this general case become so cumbersome that the result obtained does not commensurate with the input labour. We, therefore, confine ourselves to the particular cases $\alpha = 0$ and $\alpha = 1/2$.

2. Preliminaries. We need the following results.

LEMMA 2.1. Suppose that f and g are regular in U , $f(0) = g(0)$, and $g'(0) \neq 0$. Suppose further that

$$\operatorname{Re} \left(1 + \frac{zg''(z)}{g'(z)} \right) > -\frac{1}{2}, \quad z \in U. \quad (2.1)$$

Then if $f(z) \prec g(z)$ in U , we have

$$\frac{1}{z} \int_0^z f(t) dt \prec \frac{1}{z} \int_0^z g(t) dt, \quad z \in U. \quad (2.2)$$

We observe that (2.1) implies that g is close-to-convex and hence univalent in U and that the right-hand side function in (2.2) is convex in U [3]. Lemma 2.1 is due to Miller, Mocanu, and Reade [1].

The underlying idea of the following result is essentially due to Zomorvič [6] (also, see [5]).

LEMMA 2.2. *Let P be regular in U , $P(0) = 1$, and $\operatorname{Re} P(z) > 0$ in U . Let μ and λ be fixed real numbers, $-\infty < \mu < \infty$, $\lambda \geq 0$, and $|z| = r < 1$. Then*

$$\operatorname{Re} \left[\mu P(z) + \frac{zP'(z)}{P(z) + \lambda} \right] \geq \begin{cases} -\left(\sqrt{\lambda(\mu+1)} - \sqrt{(a+\lambda)}\right)^2, & \text{if } \frac{\lambda(a+\lambda)}{(a-\rho+\lambda)^2} \geq \mu+1 \\ \geq \frac{\lambda(a+\lambda)}{(a+\rho+\lambda)^2}, \\ (a-\rho)\left(\mu - \frac{\rho}{a-\rho+\lambda}\right), & \text{if } \mu+1 > \frac{\lambda(a+\lambda)}{(a-\rho+\lambda)^2}, \\ (a+\rho)\left(\mu + \frac{\rho}{a+\rho+\lambda}\right), & \text{if } \mu+1 < \frac{\lambda(a+\lambda)}{(a+\rho+\lambda)^2}, \end{cases} \quad (2.3)$$

where $a = (1+r^2)/(1-r^2)$ and $\rho = 2r/(1-r^2)$.

PROOF. Making use of the inequality (2.3) (see [5])

$$\left| zP'(z) - \frac{P^2(z) - 1}{2} \right| \leq \frac{\rho^2 - \rho_0^2}{2}, \quad (2.4)$$

where $|P(z) - a| = \rho_0 \leq \rho$, we get

$$\operatorname{Re} \left[\mu P(z) + \frac{zP'(z)}{P(z) + \lambda} \right] \geq \operatorname{Re} \left[\mu P(z) + \frac{P(z) - \lambda}{2} + \frac{(\lambda^2 - 1)(\overline{P(z)} + \lambda)}{2|P(z) + \lambda|^2} \right] - \frac{\rho^2 - \rho_0^2}{2|P(z) + \lambda|}. \quad (2.5)$$

Taking $P(z) = a + \xi + i\eta$ and $R_1^2 = (a + \xi + \lambda)^2 + \eta^2$, we get

$$\begin{aligned} \operatorname{Re} \left[\mu P(z) + \frac{zP'(z)}{P(z) + \lambda} \right] &\geq \mu(a + \xi) + \frac{a + \xi - \lambda}{2} + \frac{(\lambda^2 - 1)(a + \xi + \lambda)}{2R_1^2} - \frac{\rho^2 - \xi^2 - \eta^2}{2R_1} \\ &= S(\xi, \eta). \end{aligned} \quad (2.6)$$

Now it is easy to see that $\partial S(\xi, \eta)/\partial \eta = 0$ and $\partial^2 S(\xi, \eta)/\partial \eta^2 > 0$ at $\eta = 0$. Therefore,

$$\begin{aligned} \min_{\eta} S(\xi, \eta) &= S(\xi, 0) \\ &= \mu(a + \xi) + \frac{a + \xi - \lambda}{2} + \frac{\lambda^2 - 1}{2(a + \xi + \lambda)} - \frac{\rho^2 - \xi^2}{2(a + \xi + \lambda)} \\ &= (\mu + 1)R + \frac{\lambda(a + \lambda)}{R} - (\mu + 2)\lambda - a \\ &= L(R), \end{aligned} \quad (2.7)$$

where $R = a + \xi + \lambda$. Now, using the fact that $|R(z) - a| < \rho$, we obtain the inequality

$$a - \rho + \lambda \leq R \leq a + \rho + \lambda. \quad (2.8)$$

It is observed that at $R = R_0 = (\lambda(a + \lambda)/(\mu + 1))^{1/2}$, $\partial L(R)/\partial R = 0$ and $\partial^2 L(R)/\partial R^2 > 0$. Thus, $R = R_0$ gives the minimum value of $L(R)$ provided R_0 lies in the range of R . In view of (2.8), this is the case if the inequality

$$\frac{\lambda(a + \lambda)}{(a - \rho + \lambda)^2} \geq \mu + 1 \geq \frac{\lambda(a + \lambda)}{(a + \rho + \lambda)^2} \quad (2.9)$$

is satisfied. Thus, if (2.9) holds, we have

$$\min_R L(R) = L(R_0) = -\left(\sqrt{\lambda(\mu + 1)} - \sqrt{\lambda(a + \lambda)}\right)^2. \quad (2.10)$$

Also, it is easy to check that when $\mu + 1 > \lambda(a + \lambda)/(a - \rho + \lambda)^2$, $L(R)$ is an increasing function of R . Therefore, in this case,

$$\min_R L(R) = L(a - \rho + \lambda) = (a - \rho) \left(\mu - \frac{\rho}{a - \rho + \lambda} \right). \quad (2.11)$$

On the other hand, when $\mu + 1 < \lambda(a + \lambda)/(a + \rho + \lambda)^2$, $L(R)$ is a decreasing function of R . Therefore, in this case,

$$\min_R L(R) = L(a + \rho + \lambda) = (a + \rho) \left(\mu + \frac{\rho}{a + \rho + \lambda} \right). \quad (2.12)$$

This completes the proof of Lemma 2.2. $\square$

3. Theorems and their proofs

THEOREM 3.1. Let f be regular in U with $f(0) = 0$ and let $g \in K$. Suppose that

$$f(z) + zf'(z) \prec g(z) + zg'(z) \quad \text{in } U. \quad (3.1)$$

Then $f(z) \prec g(z)$ at least in $|z| < r_0$, where $r_0 = \sqrt{5}/3 = 0.745\dots$

PROOF. Let us take

$$h(z) = g(z) + zg'(z). \quad (3.2)$$

Since $g \in K$, we can put

$$1 + \frac{zg''(z)}{g'(z)} = P(z), \quad (3.3)$$

where $P(z)$ is regular in U , $P(0) = 1$, and $\operatorname{Re} P(z) > 0$ in U . Now, from (3.2) and (3.3), we get

$$1 + \frac{zh''(z)}{h'(z)} = P(z) + \frac{zP'(z)}{P(z) + 1}. \quad (3.4)$$

Taking $\mu = \lambda = 1$ in Lemma 2.2, we easily obtain

$$\operatorname{Re} \left(1 + \frac{zh''(z)}{h'(z)} \right) \geq \begin{cases} \frac{1-2r}{1+r}, & \text{if } 0 \leq r < \frac{3}{5}, \\ -2 \left[1 - \frac{a}{\sqrt{1-r^2}} \right]^2, & \text{if } \frac{3}{5} \leq r < 1, \end{cases} \quad (3.5)$$

where $|z| = r < 1$. Now, it is easy to verify that for $0 \leq r < 3/5$, $\operatorname{Re}(1 + zh''(z)/h'(z)) > -1/2$ and for $3/5 \leq r < 1$, $\operatorname{Re}(1 + zh''(z)/h'(z)) > -1/2$ whenever $9r^4 + 22r^2 - 15 < 0$ or whenever $r < r_0$, where $r_0 = \sqrt{5}/3$ is the smallest positive root of $9r^4 + 22r^2 - 15 = 0$. The assertion of our theorem now follows from Lemma 2.1. $\square$

THEOREM 3.2. *Let f be regular in U with $f(0) = 0$ and let $g \in K(1/2)$. Suppose that*

$$f(z) + zf'(z) \prec g(z) + zg'(z) \quad \text{in } U. \quad (3.6)$$

Then

$$f(z) \prec g(z) \quad (3.7)$$

at least in $|z| < r_1$, where $r_1 = ((51 - 24\sqrt{2})/23)^{1/2} = 0.8612\dots$

PROOF. Let us put

$$h(z) = g(z) + zg'(z). \quad (3.8)$$

Since $g \in K(1/2)$, we can write

$$1 + \frac{zg''(z)}{g'(z)} = \frac{P(z) + 1}{2}, \quad (3.9)$$

where $P(z)$ is regular in U , $P(0) = 1$, and $\operatorname{Re} P(z) > 0$ in U . From (3.8) and (3.9), we obtain

$$1 + \frac{zh''(z)}{h'(z)} = \frac{1}{2} + \frac{P(z)}{2} + \frac{zP'(z)}{P(z) + 3}. \quad (3.10)$$

Using Lemma 2.2 (with $\mu = 1/2$ and $\lambda = 3$), we obtain, after some calculations,

$$\operatorname{Re} \left[1 + \frac{zh''(z)}{h'(z)} \right] \geq \begin{cases} \frac{2}{(1+r)(2+r)}, & \text{if } 0 \leq r < \frac{-1+\sqrt{5}}{2}, \\ 6 \left[\frac{2-r^2}{1-r^2} \right]^{1/2} - 2 \left(\frac{4-3r^2}{1-r^2} \right), & \text{if } \frac{-1+\sqrt{5}}{2} \leq r < 1, \end{cases} \quad (3.11)$$

where $|z| = r < 1$.

Now, we can easily check that for $0 \leq r < (-1 + \sqrt{5})/2$, $\operatorname{Re}(1 + zh''(z)/h'(z)) > -1/2$ and for $(-1 + \sqrt{5})/2 \leq r < 1$, $\operatorname{Re}(1 + zh''(z)/h'(z)) > -1/2$ whenever $23r^4 - 102r^2 + 63 > 0$ or whenever $r < r_1$, where $r_1 = ((51 - 24\sqrt{2})/23)^{1/2}$ is the smallest positive root of $23r^4 - 102r^2 + 63 = 0$. The desired result now follows from Lemma 2.1. $\square$

In the following theorem, we take for g some distinguished members of K .

THEOREM 3.3. *Let f be regular in U with $f(0) = 0$ and let $f(z) + zf'(z) \prec g(z) + zg'(z)$ in U . Then*

- (a) $f(z) \prec g(z)$ in U if $g(z) = z/(1-z)$;

(b) $f(z) \prec g(z)$ at least in $|z| < \rho_1 = ((28 - 8\sqrt{7})/7)^{1/2} = 0.98\dots$ if $g(z) = -\log(1-z)$;

(c) $f(z) \prec g(z)$ in U if $g(z) = z + \lambda z^2$, $|\lambda| \leq 1/5$;

(d) $f(z) \prec g(z)$ at least in $|z| < \rho_2 = (9 - \sqrt{33})/4 = 0.8138\dots$ if $g(z) = e^z - 1$.

We observe that the functions g defined in (a), (b), (c), and (d) belong to K , $K(1/2)$, $K(1/3)$, and K , respectively.

PROOF. We omit the proofs of parts (a), (c), and (d) and proceed to prove part (b). Let $h(z) = g(z) + zg'(z)$, where $g(z) = -\log(1-z)$. Then $h(0) = 0$ and $h'(0) \neq 0$. A simple computation shows that the condition

$$\operatorname{Re} \left(1 + \frac{zh''(z)}{h'(z)} \right) > -\frac{1}{2} \quad (3.12)$$

is equivalent to

$$\operatorname{Re} \left[\frac{2}{(1-z)(2-z)} + \frac{1}{2} \right] > 0. \quad (3.13)$$

If we let $z = re^{i\theta}$, $0 \leq r < 1$ and $0 \leq \theta \leq 2\pi$, then condition (3.13) takes the form

$$\varphi(x) = 16r^2x^2 - 6r(4+r^2)x + r^4 + r^2 + 12 > 0, \quad (3.14)$$

where $x = \cos \theta$, $0 \leq \theta \leq 2\pi$. For $r = 0$, (3.14) is obviously satisfied. We, therefore, let $r \neq 0$. Now, it can be readily verified that at $x = x_0 = (12 + 3r^2)/16r$, we have $\varphi'(x) = 0$ and $\varphi''(x) > 0$.

Thus, $x = x_0$ gives the minimum value of $\varphi(x)$ provided $-1 \leq x_0 \leq 1$. This is true if $r \geq \rho_0 = (8 - \sqrt{28})/3 = 0.9028\dots$. Therefore, for $r \in [\rho_0, 1]$,

$$\min_{x \in [-1, 1]} \varphi(x) = \varphi(x_0) = \frac{7r^4 - 56r^2 + 48}{16}. \quad (3.15)$$

Hence, in this case, (3.14) is satisfied if $7r^4 - 56r^2 + 48 > 0$, i.e., if $r < \rho_1 = ((28 - 8\sqrt{7})/7)^{1/2} = 0.98\dots$. Also, for $r \in [0, \rho_0)$, we can easily verify that $\varphi(x)$ is a decreasing function of x . Hence, in this case,

$$\begin{aligned} \min_{x \in [-1, 1]} \varphi(x) &= \varphi(1) = 4 - 6^3 + 17^2 - 24 + 12 \\ &= (1-r)(2-r)(r^2 - 3r + 6) > 0. \end{aligned} \quad (3.16)$$

Therefore, we conclude that for $0 \leq r < \rho_1$,

$$\operatorname{Re} \left(1 + \frac{zh''(z)}{h'(z)} \right) > -\frac{1}{2}. \quad (3.17)$$

Conclusion (b) now follows in view of Lemma 2.1. $\square$

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