AN APPLICATION OF ALMOST INCREASING SEQUENCES

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ABSTRACT. We extended a theorem of Mishra and Srivastava (1983–1984) on $|C, 1|_k$ summability factors, using almost increasing sequences, to $|\bar{N}, p_n|_k$ summability under weaker conditions.

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Let $\sum a_n$ be a given infinite series with partial sums (s_n) . We denote by z_n the n th $(C, 1)$ mean of the sequence (s_n) . The series $\sum a_n$ is said to be summable $|C, 1|_k$, $k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty} n^{k-1} |z_n - z_{n-1}|^k < \infty. \quad (1)$$

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^n p_v \rightarrow \infty, \quad \text{as } n \rightarrow \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1). \quad (2)$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{v=0}^n p_v s_v \quad (3)$$

defines the sequence (t_n) of the $(\bar{N}, p_n)$ mean of the sequence (s_n) , generated by the sequence of coefficients (p_n) (see [3]). The series $\sum a_n$ is said to be summable $|\bar{N}, p_n|_k$, $k \geq 1$, if (see [1])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} |t_n - t_{n-1}|^k < \infty. \quad (4)$$

In the special case when $p_n = 1$ for all values of n (resp., $p_n = 1/(n+1)$), $|\bar{N}, p_n|_k$ summability is the same as $|C, 1|_k$ (resp., $|\bar{N}, 1/(n+1)|_k$) summability.

Concerning the $|C, 1|_k$ summability factors the following theorem is known.

THEOREM 1 (see [4]). *Let (X_n) be a positive nondecreasing sequence and let (β_n) and (λ_n) be sequences such that*

$$|\Delta \lambda_n| \leq \beta_n, \quad (5)$$

$$\beta_n \rightarrow 0, \quad \text{as } n \rightarrow \infty, \quad (6)$$

$$\sum_{n=1}^{\infty} n |\Delta \beta_n| X_n < \infty, \quad (7)$$

$$|\lambda_n| X_n = O(1), \quad \text{as } n \rightarrow \infty. \quad (8)$$

If

$$\sum_{n=1}^m \frac{1}{n} |s_n|^k = O(X_m), \quad \text{as } m \rightarrow \infty, \quad (9)$$

then the series $\sum a_n \lambda_n$ is summable $|C, 1|_k$, $k \geq 1$.

The aim of this paper is to extend Theorem 1 to $|\bar{N}, p_n|_k$ summability under weaker conditions. Thus we need the concept of almost increasing sequence. A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$. Obviously every increasing sequence is almost increasing but the converse need not be true as can be seen from the example $b_n = ne^{(-1)^n}$.

Now, we shall prove the following theorem.

THEOREM 2. *Let (X_n) be an almost increasing sequence and let the condition (9) of Theorem 1 be satisfied. If the sequences (β_n) and (λ_n) such that conditions (5), (6), (7), and (8) of Theorem 1 are satisfied. If (p_n) is a sequence such that*

$$\sum_{n=1}^m \frac{p_n}{P_n} |s_n|^k = O(X_m), \quad \text{as } m \rightarrow \infty, \quad (10)$$

then the series $\sum a_n \lambda_n$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.

We need the following lemma for the proof of our theorem.

LEMMA 3. *Under the conditions on (X_n) , (β_n) , and (λ_n) as taken in the statement of the theorem, the following conditions hold, when (7) is satisfied,*

$$n\beta_n X_n = O(1), \quad \text{as } n \rightarrow \infty, \quad (11)$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \quad (12)$$

PROOF. Let $Ac_n \leq X_n \leq Bc_n$, where (c_n) is an increasing sequence. In this case,

$$\begin{aligned} nX_n\beta_n &\leq nBc_n \left| \sum_{v=n}^{\infty} \Delta\beta_v \right| \leq nBc_n \sum_{v=n}^{\infty} |\Delta\beta_v| \\ &\leq B \sum_{v=n}^{\infty} v c_v |\Delta\beta_v| \leq \frac{B}{A} \sum_{v=n}^{\infty} v |\Delta\beta_v| X_v. \end{aligned} \quad (13)$$

Hence $n\beta_n X_n = O(1)$ as $n \rightarrow \infty$. Again

$$\begin{aligned}
\sum_{n=1}^{\infty} X_n \beta_n &\leq B \sum_{n=1}^{\infty} c_n \beta_n = B \sum_{n=1}^{\infty} c_n \left| \sum_{v=n}^{\infty} \Delta \beta_v \right| \\
&\leq B \sum_{n=1}^{\infty} c_n \sum_{v=n}^{\infty} |\Delta \beta_v| = B \sum_{v=1}^{\infty} |\Delta \beta_v| \sum_{n=1}^v c_n \\
&\leq B \sum_{v=1}^{\infty} v c_v |\Delta \beta_v| \leq \frac{B}{A} \sum_{v=1}^{\infty} v X_v |\Delta \beta_v| < \infty.
\end{aligned} \tag{14}$$

Hence $\sum_{n=1}^{\infty} X_n \beta_n < \infty$. $\square$

PROOF OF THE THEOREM. Let (T_n) be the sequence of $(\tilde{N}, p_n)$ mean of the series $\sum a_n \lambda_n$. Then, by definition, we have

$$T_n = \frac{1}{p_n} \sum_{v=0}^n p_v \sum_{i=0}^v a_i \lambda_i = \frac{1}{p_n} \sum_{v=0}^n (p_n - p_{v-1}) a_v \lambda_v. \tag{15}$$

Then, for $n \geq 1$, we have

$$T_n - T_{n-1} = \frac{p_n}{p_n p_{n-1}} \sum_{v=1}^n p_{v-1} a_v \lambda_v. \tag{16}$$

By Abel's transformation, we have

$$\begin{aligned}
T_n - T_{n-1} &= -\frac{p_n}{p_n p_{n-1}} \sum_{v=1}^{n-1} \Delta(p_{v-1} \lambda_v) s_v + \frac{p_n s_n \lambda_n}{p_n} \\
&= \frac{p_n}{p_n p_{n-1}} \sum_{v=1}^{n-1} p_v s_v \lambda_v - \frac{p_n}{p_n p_{n-1}} \sum_{v=1}^{n-1} p_v s_v \Delta \lambda_v + \frac{p_n s_n \lambda_n}{p_n} \\
&= T_{n,1} + T_{n,2} + T_{n,3},
\end{aligned} \tag{17}$$

let us denote the three terms in (17) by $T_{n,1}$, $T_{n,2}$, and $T_{n,3}$.

Since

$$|T_{n,1} + T_{n,2} + T_{n,3}|^k \leq 3^k (|T_{n,1}|^k + |T_{n,2}|^k + |T_{n,3}|^k), \tag{18}$$

to complete the proof of the theorem, it is sufficient to show that

$$\sum_{n=1}^{\infty} \left(\frac{p_n}{p_n} \right)^{k-1} |T_{n,r}|^k < \infty, \quad \text{for } r = 1, 2, 3. \tag{19}$$

Since $|\lambda_n| = O(1/X_n) = O(1)$, by (8), applying Hölder's inequality with indices k and k' , where $(1/k) + (1/k') = 1$ and $k > 1$, we get

$$\begin{aligned}
\sum_{n=2}^{m+1} \left(\frac{p_n}{p_n} \right)^{k-1} |T_{n,1}|^k &\leq \sum_{n=2}^{m+1} \frac{p_n}{p_n p_{n-1}^k} \left(\sum_{v=1}^{n-1} p_v |\lambda_v| |s_v| \right)^k \\
&\leq \sum_{n=2}^{m+1} \frac{p_n}{p_n p_{n-1}^k} \left(\sum_{v=1}^{n-1} p_v |\lambda_v|^k |s_v|^k \right) \left(\frac{1}{p_{n-1}} \sum_{v=1}^{n-1} p_v \right)^{k-1} \\
&= O(1) \sum_{v=1}^m p_v |\lambda_v|^k |s_v|^k \sum_{n=v+1}^{m+1} \frac{p_n}{p_n p_{n-1}^k} \\
&= O(1) \sum_{v=1}^m \frac{p_v}{p_v} |\lambda_v|^k |s_v|^k \\
&= O(1) \sum_{v=1}^m \frac{p_v}{p_v} |\lambda_v| |\lambda_v|^{k-1} |s_v|^k \\
&= O(1) \sum_{v=1}^m \frac{p_v}{p_v} |\lambda_v| |s_v|^k \\
&= O(1) \sum_{v=1}^{m-1} \Delta |\lambda_v| \sum_{i=1}^v \frac{p_i}{p_i} |s_i|^k + O(1) |\lambda_m| \sum_{v=1}^m \frac{p_v}{p_v} |s_v|^k \\
&= O(1) \sum_{v=1}^{m-1} |\Delta \lambda_v| X_v + O(1) |\lambda_m| X_m \\
&= O(1), \quad \text{as } m \rightarrow \infty,
\end{aligned} \tag{20}$$

by virtue of (5), (8), (10), and (12). Again applying Hölder's inequality, as in $T_{n,1}$, we have

$$\begin{aligned}
\sum_{n=2}^{m+1} \left(\frac{p_n}{p_n} \right)^{k-1} |T_{n,2}|^k &\leq \sum_{n=2}^{m+1} \frac{p_n}{p_n p_{n-1}^k} \left(\sum_{v=1}^{n-1} p_v |\Delta \lambda_v| |s_v| \right)^k \\
&\leq \sum_{n=2}^{m+1} \frac{p_n}{p_n p_{n-1}^k} \left(\sum_{v=1}^{n-1} p_v |s_v| \beta_v \right)^k \\
&\leq \sum_{n=2}^{m+1} \frac{p_n}{p_n p_{n-1}^k} \sum_{v=1}^{n-1} p_v |s_v|^k \beta_v \left(\frac{1}{p_{n-1}} \sum_{v=1}^{n-1} p_v \beta_v \right)^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \frac{p_n}{p_n p_{n-1}^k} \sum_{v=1}^{n-1} p_v |s_v|^k \beta_v \\
&= O(1) \sum_{v=1}^m p_v |s_v|^k \beta_v \sum_{n=v+1}^{m+1} \frac{p_n}{p_n p_{n-1}^k} \\
&= O(1) \sum_{v=1}^m |s_v|^k \beta_v \\
&= O(1) \sum_{v=1}^m v \beta_v \frac{1}{v} |s_v|^k \\
&= O(1) \sum_{v=1}^{m-1} \Delta(v \beta_v) \sum_{u=1}^v \frac{1}{u} |s_u|^k + O(1) m \beta_m \sum_{v=1}^m \frac{1}{v} |s_v|^k
\end{aligned}$$

$$\begin{aligned}
&= O(1) \sum_{v=1}^{m-1} |\Delta(v\beta_v)| X_v + O(1)m\beta_m X_m \\
&= O(1) \sum_{v=1}^{m-1} v |\Delta\beta_v| X_v + O(1) \sum_{v=1}^{m-1} \beta_{v+1} X_v + O(1)m\beta_m X_m \\
&= O(1), \quad \text{as } m \rightarrow \infty,
\end{aligned} \tag{21}$$

by virtue of (5), (7), (9), (11), and (12). Finally, as in $T_{n,1}$, we have that

$$\sum_{n=1}^m \left(\frac{p_n}{p_n} \right)^{k-1} |T_{n,3}|^k = O(1) \sum_{n=1}^m \frac{p_n}{p_n} |\lambda_n| |s_n|^k = O(1), \quad \text{as } m \rightarrow \infty. \tag{22}$$

Therefore we get (19). This completes the proof of the theorem. $\square$

It should be noted that if we take (X_n) is a positive nondecreasing sequence and $p_n = 1$ for all values of n in this theorem, then we get Theorem 1. In this case the condition (10) reduces to the condition (9). Also, if we take $p_n = 1/(n+1)$ in this theorem, then we get a result concerning the $|\tilde{N}, 1/(n+1)|_k$ summability factors.

REFERENCES

- [1] H. Bor, *A note on two summability methods*, Proc. Amer. Math. Soc. **98** (1986), no. 1, 81-84. MR 87i:40007. Zbl 601.40004.
- [2] T. M. Flett, *On an extension of absolute summability and some theorems of Littlewood and Paley*, Proc. London Math. Soc. (3) **7** (1957), 113-141. MR 19,266a. Zbl 109.04402.
- [3] G. H. Hardy, *Divergent Series*, Clarendon Press, Oxford, 1949. MR 11,25a. Zbl 032.05801.
- [4] K. N. Mishra and R. S. L. Srivastava, *On absolute Cesàro summability factors of infinite series*, Portugal. Math. **42** (1983/84), no. 1, 53-61 (1985). MR 87a:40003. Zbl 597.40003.

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