ON A CLASS OF CONTACT RIEMANNIAN MANIFOLDS

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ABSTRACT. We determine a locally symmetric or a Ricci-parallel contact Riemannian manifold which satisfies a $D$-homothetically invariant condition.

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1. Introduction. In [8] Tanno proved that a locally symmetric $K$-contact Riemannian manifold is of constant curvature $1$, which generalizes the corresponding result for a Sasakian manifold due to Okumura [6]. For dimensions greater than or equal to $5$ it was proved by Olszak [7] that there are no contact Riemannian structures of constant curvature unless the constant is $1$ and in which case the structure is Sasakian. Further, Blair and Sharma [4] proved that a $3$-dimensional locally symmetric contact Riemannian manifold is either flat or is Sasakian and of constant curvature $1$. By the recent result [5] and private communication with Blair we know that the simply connected covering space of a complete $5$-dimensional locally symmetric contact Riemannian manifold is either $S^5(1)$ or $E^3 \times S^2(4)$. The question of the classification of locally symmetric contact Riemannian manifolds in higher dimensions is still open.

On the other hand, recently, Blair, Koufogiorgos and Papantoniou [3] introduced a class of contact Riemannian manifolds which is characterized by the equation

$$R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY),$$

(1.1)

where $\kappa, \mu$ are constant and $2h$ is the Lie derivative of $\phi$ in the direction $\xi$. It is remarkable that this class of spaces is invariant under $D$-homothetic deformations (see [3]). It was also proved in [3] that a Sasakian manifold, in particular, is determined by $\kappa = 1$ and further that this class contains the tangent sphere bundle of Riemannian manifolds of constant curvature. In this paper, we determine a locally symmetric or a Ricci-parallel contact Riemannian manifold which satisfies (1.1). More precisely, we prove the following two Theorems 1.1 and 1.2 in Sections 3 and 4.

**Theorem 1.1.** Let $M$ be a contact Riemannian manifold satisfying (1.1). Suppose that $M$ is locally symmetric. Then $M$ is the product of flat $(n+1)$-dimensional manifold and an $n$-dimensional manifold of positive constant curvature equal to $4$, or a space of constant curvature $1$ and in which case the structure is Sasakian.

**Theorem 1.2.** Let $M$ be a contact Riemannian manifold satisfying (1.1). Suppose that $M$ is Ricci-parallel. Then $M$ is the product of flat $(n+1)$-dimensional manifold and
an $n$-dimensional manifold of positive constant curvature equal to 4 or an Einstein-Sasakian manifold.

2. Preliminaries. All manifolds in the present paper are assumed to be connected and of class $C^\infty$. A $(2n+1)$-dimensional manifold $M^{2n+1}$ is said to be a contact manifold if it admits a global 1-form $\eta$ such that $\eta \wedge (d\eta)^n \neq 0$ everywhere. Given a contact form $\eta$, we have a unique vector field $\xi$, which is called the characteristic vector field, satisfying $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$ for any vector field $X$. It is well known that there exists an associated Riemannian metric $g$ and a $(1,1)$-type tensor field $\phi$ such that

$$\eta(X) = g(X, \xi), \quad d\eta(X, Y) = g(X, \phi Y), \quad \phi^2 X = -X + \eta(X)\xi,$$  \hspace{1cm} (2.1)

where $X$ and $Y$ are vector fields on $M$. From (2.1) it follows that

$$\phi \xi = 0, \quad \eta \circ \phi = 0, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$  \hspace{1cm} (2.2)

A Riemannian manifold $M$ equipped with structure tensors $(\eta, g)$ satisfying (2.1) is said to be a contact Riemannian manifold and is denoted by $M = (M; \eta, g)$. Given a contact Riemannian manifold $M$, we define a $(1,1)$-type tensor field $h$ by $h = L_\xi \phi/2$, where $L$ denotes Lie differentiation. Then we may observe that $h$ is symmetric and satisfies

$$h \xi = 0, \quad h = -\phi h, \hspace{1cm} (2.3)$$

$$\nabla_X \xi = -\phi X - \phi hX, \hspace{1cm} (2.4)$$

where $\nabla$ is Levi-Civita connection. From (2.3) and (2.4), we see that each trajectory of $\xi$ is a geodesic.

A contact Riemannian manifold for which $\xi$ is Killing is called a $K$-contact Riemannian manifold. It is easy to see that a contact Riemannian manifold is $K$-contact if and only if $h = 0$. For a contact Riemannian manifold $M$ one may define naturally an almost complex structure $J$ on $M \times \mathbb{R}$;

$$J \left( X, f \frac{d}{dt} \right) = \left( \phi X - f \xi, \eta(X) \frac{d}{dt} \right),$$  \hspace{1cm} (2.5)

where $X$ is a vector field tangent to $M$, $t$ the coordinate of $\mathbb{R}$, and $f$ a function on $M \times \mathbb{R}$. If the almost complex structure $J$ is integrable, $M$ is said to be normal or Sasakian. It is known that $M$ is normal if and only if $M$ satisfies

$$[\phi, \phi] + 2d\eta \otimes \xi = 0,$$  \hspace{1cm} (2.6)

where $[\phi, \phi]$ is the Nijenhuis torsion of $\phi$. A Sasakian manifold is characterized by a condition

$$(\nabla_X \phi) Y = g(X, Y)\xi - \eta(Y)X$$  \hspace{1cm} (2.7)

for all vector fields $X$ and $Y$ on the manifold. We denote by $R$ the Riemannian curvature tensor of $M$ defined by

$$R(X, Y)Z = \nabla_X (\nabla_Y Z) - \nabla_Y (\nabla_X Z) - \nabla_{[X,Y]} Z$$  \hspace{1cm} (2.8)
for all vector fields $X, Y, Z$ on $M$. It is well known that $M$ is Sasakian if and only if

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y$$

(2.9)

for all vector fields $X$ and $Y$. For a contact Riemannian manifold $M$, the tangent space $T_p M$ at each point $p \in M$ is decomposed as $T_p M = D_p \oplus \{\xi\}_p$ (direct sum), where we denote $D_p = \{v \in T_p M \mid \eta(v) = 0\}$. Then $D : p \to D_p$ defines a distribution orthogonal to $\xi$. The $2n$-dimensional distribution $D$ is called the contact distribution. A contact Riemannian manifold is said to be $\eta$-Einstein if

$$Q = aI + b\eta \otimes \xi,$$

(2.10)

where $Q$ is the Ricci operator and $a, b$ are smooth functions on $M$.

For more details about the fundamental properties on contact Riemannian manifolds we refer to [1, 2]. Blair [2] proved the following theorem.

**Theorem 2.1.** Let $M = (M; \eta, g)$ be a contact Riemannian manifold and suppose that $R(X, Y)\xi = 0$ for all vector fields $X, Y$ on $M$. Then $M$ is locally the product of $(n + 1)$-dimensional flat manifold and an $n$-dimensional manifold of positive constant curvature 4.

Recently, Blair, Koufogiorgos, and Papantoniou [3] introduced a class of contact Riemannian manifolds which are characterized by equation (1.1). A $D$-homothetic deformation (cf. [9]) is defined by a change of structure tensors of the form

$$\bar{\eta} = a\eta, \quad \bar{\xi} = \frac{1}{a}\xi, \quad \bar{\phi} = \phi, \quad \bar{g} = ag + a(a - 1)\eta \otimes \eta,$$

(2.11)

where $a$ is a positive constant. It was shown that [3] a contact Riemannian manifold $M$ satisfying (1.1) is obtained by applying a $D$-homothetic deformation on a contact Riemannian manifold with $R(X, Y)\xi = 0$ and that the property (1.1) is invariant under the $D$-homothetic deformation. It is well known that the tangent sphere bundle of a flat Riemannian manifold admits a contact Riemannian structure satisfying $R(X, Y)\xi = 0$ [1, page 137]. In [3] the authors classified the 3-dimensional case and showed that this class contains the tangent sphere bundles of Riemannian manifolds of constant sectional curvature. Furthermore in the same paper they showed that $M$ satisfies

$$(\nabla_Z h)X = (1 - \kappa)[(1 - \kappa)g(Z, \phi X) + g(Z, h\phi X)]\xi + \eta(X)(h\phi + h\phi h)Z - \mu\eta(Z)\phi h X$$

(2.12)

for any vector fields $X, Z$ on $M$. Here, we state some useful results in [3] to prove our Theorems 1.1 and 1.2.

**Proposition 2.2.** Let $M = (M; \eta, g)$ be a contact Riemannian manifold which satisfies (1.1), where $\kappa < 1$.

(i) If $X, Y \in D(\lambda)$ (respectively, $D(-\lambda)$), then $\nabla_X Y \in D(\lambda)$ (respectively, $D(-\lambda)$).

(ii) If $X \in D(\lambda)$, $Y \in D(-\lambda)$, then $\nabla_X Y$ (respectively, $\nabla_Y X$) $\in D(-\lambda) \oplus D(0)$ (respectively, $D(\lambda) \oplus D(0)$).
Theorem 2.3. Let $M = (M; \eta, g)$ be a contact Riemannian manifold which satisfies (1.1), then $\kappa \leq 1$. If $\kappa = 1$, then $h = 0$ and $M$ is a Sasakian manifold. If $k < 1$, then $M$ admits three mutually orthogonal and integrable distributions $D(0), D(\lambda)$, and $D(-\lambda)$, defined by the eigenspaces of $h$, where $\lambda = \sqrt{1 - \kappa}$. Moreover

\begin{align*}
R(X_{\lambda}, Y_{\lambda}) Z_{-\lambda} &= (\kappa - \mu) \{ g(\phi Y_{\lambda}, Z_{-\lambda}) \phi X_{\lambda} - g(\phi X_{\lambda}, Z_{-\lambda}) \phi Y_{\lambda} \}, \\
R(X_{-\lambda}, Y_{\lambda}) Z_{\lambda} &= (\kappa - \mu) \{ g(\phi Y_{-\lambda}, Z_{\lambda}) \phi X_{-\lambda} - g(\phi X_{-\lambda}, Z_{\lambda}) \phi Y_{\lambda} \}, \\
R(X_{\lambda}, Y_{-\lambda}) Z_{\lambda} &= \kappa g(\phi X_{\lambda}, Z_{\lambda}) \phi Y_{-\lambda} + \mu g(\phi X_{\lambda}, Y_{-\lambda}) \phi Z_{-\lambda}, \\
R(X_{\lambda}, Y_{\lambda}) Z_{\lambda} &= -\kappa g(\phi Y_{-\lambda}, Z_{\lambda}) \phi X_{\lambda} - \mu g(\phi Y_{-\lambda}, X_{\lambda}) \phi Z_{\lambda}, \\
R(X_{-\lambda}, Y_{\lambda}) Z_{\lambda} &= [2(1 + \lambda) - \mu] \{ g(Y_{\lambda}, Z_{\lambda}) X_{\lambda} - g(X_{\lambda}, Z_{\lambda}) Y_{\lambda} \}, \\
R(X_{-\lambda}, Y_{-\lambda}) Z_{\lambda} &= [2(1 - \lambda) - \mu] \{ g(Y_{-\lambda}, Z_{-\lambda}) X_{-\lambda} - g(X_{-\lambda}, Z_{-\lambda}) Y_{\lambda} \},
\end{align*}

where $X_{\lambda}, Y_{\lambda}, Z_{\lambda} \in D(\lambda)$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in D(-\lambda)$.

Theorem 2.4. For a contact Riemannian manifold satisfying (1.1) with $\kappa < 1$, the Ricci operator $Q$ is given by

\begin{equation}
Q = [2(n - 1) - n\mu] I + [2(n - 1) + \mu] h + [2(1 - n) + n(2\kappa + \mu)] \eta \otimes \xi. \tag{2.14}
\end{equation}

For more results about a contact Riemannian manifold satisfying (1.1), we refer to [3].

3. Proof of Theorem 1.1. Let $M^{2n+1}$ be a $(2n + 1)$-dimensional contact Riemannian manifold which satisfies (1.1). Suppose that $M$ is locally symmetric, that is, $\nabla R = 0$. In view of the results of the Sasakian case [6] and the 3-dimensional contact Riemannian case [4], we now assume that $n > 1$ and $M$ is non-Sasakian ($\kappa \neq 1$). From $h\xi = 0$, with (2.4) we have

\begin{equation}
(\nabla h)\xi = \nabla (h\xi) - h\nabla \xi = (h\phi + h\phi h)Z. \tag{3.1}
\end{equation}

If we differentiate (1.1) covariantly, then using (2.4) we get

\begin{align*}
R(X, Y)(-\phi Z - \phi hZ) &= \kappa \{ g(-\phi Z - \phi hZ, Y) X - g(-\phi Z - \phi hZ, X) Y \} \\
&\quad + \mu \{ g(-\phi Z - \phi hZ, Y) hX + \eta(Y)(\nabla h)X \\
&\quad - g(-\phi Z - \phi hZ, X) hY - \eta(X)(\nabla h)Y \} \tag{3.2}
\end{align*}

for any vector fields $X, Y$ on $M$. Putting $Y = \xi$, then with (2.2), (2.3), and (3.1) we have

\begin{equation}
R(X, \xi)(-\phi Z - \phi hZ) = \kappa g(\phi Z + \phi hZ, X) \xi + \mu \{ (\nabla h)X - \eta(X)(h\phi + h\phi h)Z \}. \tag{3.3}
\end{equation}

Together with (1.1) we have

\begin{equation}
\mu(\nabla h)X = \mu \{ \eta(X)(h\phi + h\phi h)Z + g((h\phi + h\phi h)Z, X) \xi \}. \tag{3.4}
\end{equation}

From (2.12) and (3.4) we have

\begin{align*}
\mu \{ \eta(X)(h\phi + h\phi h)Z + g((h\phi + h\phi h)Z, X) \xi \} \\
= \mu \{ (1 - \kappa) (\eta(Z, \phi X) + g(Z, h\phi X)) \xi + \eta(X)(h\phi + h\phi h)Z - \mu \eta(Z) \phi hX \} \tag{3.5}
\end{align*}
for any vector fields $X, Z$ in $M$. If we put $Z = \xi$, then we have
\[
\mu^2 \phi h X = 0. \tag{3.6}
\]
Since $M$ is not Sasakian, we have $\mu = 0$. Now, we consider the following equation in Theorem 2.3:
\[
R(X_{\lambda}, Y_{\lambda}) Z_{\lambda} = 2(1 + \lambda) \{ g(Y_{\lambda}, Z_{\lambda}) X_{\lambda} - g(X_{\lambda}, Z_{\lambda}) Y_{\lambda} \}, \tag{3.7}
\]
where $X_{\lambda}, Y_{\lambda}, Z_{\lambda} \in D(\lambda)$. Differentiating (3.7) covariantly with respect to $V_{-\lambda} \in D(\lambda)$, then since $M$ is locally symmetric we have
\[
R(\nabla_{V_{-\lambda}} X_{\lambda}, Y_{\lambda}) Z_{\lambda} + R(X_{\lambda}, \nabla_{V_{-\lambda}} Y_{\lambda}) Z_{\lambda} + R(X_{\lambda}, Y_{\lambda}) \nabla_{V_{-\lambda}} Z_{\lambda}
= 2(1 + \lambda) \{ g(\nabla_{V_{-\lambda}} X_{\lambda}, Z_{\lambda}) X_{\lambda} + g(Y_{\lambda}, \nabla_{V_{-\lambda}} Z_{\lambda}) X_{\lambda} + g(Y_{\lambda}, Z_{\lambda}) \nabla_{V_{-\lambda}} X_{\lambda} \}
- g(\nabla_{V_{-\lambda}} X_{\lambda}, Z_{\lambda}) Y_{\lambda} - g(X_{\lambda}, \nabla_{V_{-\lambda}} Z_{\lambda}) Y_{\lambda} - g(X_{\lambda}, Z_{\lambda}) \nabla_{V_{-\lambda}} Y_{\lambda}. \tag{3.8}
\]
Together with Proposition 2.2 and using (3.7) again we get
\[
g(\nabla_{V_{-\lambda}} X_{\lambda}, \xi) R(\xi, Y_{\lambda}) Z_{\lambda} + g(\nabla_{V_{-\lambda}} Y_{\lambda}, \xi) R(X_{\lambda}, \xi) Z_{\lambda} + g(\nabla_{V_{-\lambda}} Z_{\lambda}, \xi) R(X_{\lambda}, Y_{\lambda}) \xi
= 2(1 + \lambda) \{ g(Y_{\lambda}, Z_{\lambda}) g(\nabla_{V_{-\lambda}} X_{\lambda}, \xi) \xi - g(X_{\lambda}, Z_{\lambda}) g(\nabla_{V_{-\lambda}} Y_{\lambda}, \xi) \xi \}. \tag{3.9}
\]
From (1.1), by using the property of the curvature tensor, we get
\[
R(\xi, X) Y = \kappa (g(Y, X) \xi - \eta(Y) X) + \mu (g(h Y, X) \xi - \eta(Y) h X). \tag{3.10}
\]
By using (1.1), (2.1), and (3.10) we have
\[
(\kappa - 2\lambda - 2) \{ g(Y_{\lambda}, Z_{\lambda}) g(X_{\lambda}, \phi V_{-\lambda} + \phi h V_{-\lambda}) \xi - g(X_{\lambda}, Z_{\lambda}) g(Y_{\lambda}, \phi V_{-\lambda} + \phi h V_{-\lambda}) \xi \} = 0, \tag{3.11}
\]
and thus we have
\[
(1 - \lambda)(\kappa - 2\lambda - 2) \{ g(Y_{\lambda}, Z_{\lambda}) g(X_{\lambda}, \phi V_{-\lambda}) \xi - g(X_{\lambda}, Z_{\lambda}) g(Y_{\lambda}, \phi V_{-\lambda}) \xi \} = 0. \tag{3.12}
\]
We may take an adapted orthonormal basis $\{\xi, e_i, \phi e_i\}$ such that $h \xi = 0$, $h e_i = \lambda_i e_i$ and $h \phi e_i = -\lambda_i \phi e_i$, $i = 1, 2, \ldots, n$ at any point $p \in M$. Since $g(\phi e_i, \phi V_{-\lambda}) = 0$ and $g(Y_{\lambda}, \xi) g(\xi, \phi V_{-\lambda}) = 0$, from (3.12) we have
\[
(1 - \lambda)(\kappa - 2\lambda - 2) \left\{ \sum_{i=1}^{n} g(Y_{\lambda}, e_i) g(e_i, \phi V_{-\lambda}) \xi 
+ \sum_{i=1}^{n} g(Y_{\lambda}, \phi e_i) g(\phi e_i, \phi V_{-\lambda}) \xi + g(Y_{\lambda}, \xi) g(\xi, \phi V_{-\lambda}) \xi 
- \sum_{i=1}^{n} g(e_i, e_i) g(Y_{\lambda}, \phi V_{-\lambda}) \xi \right\} = 0. \tag{3.13}
\]
And hence, we obtain
\[
(1 - n)(1 - \lambda)(\kappa - 2\lambda - 2) g(Y_{\lambda}, \phi V_{-\lambda}) \xi = 0. \tag{3.14}
\]
If we put $\phi V_\lambda = Y_\lambda$ in (3.14), then it follows that
\[(1 - n)(1 - \lambda)(\kappa - 2\lambda - 2) = 0, \tag{3.15}\]
where $X, Y$ are vector fields on $M$. Since $n > 1$ and $\kappa = 1 - \lambda^2$, we conclude that $\kappa = \mu = 0$, that is, $M$ satisfies $R(X, Y)\xi = 0$ for any vector fields $X, Y$ in $M$. Therefore by the results in [4, 6] and Theorem 2.1 we have proved Theorem 1.1.

4. Proof of Theorem 1.2. Let $M$ be a contact Riemannian manifold which satisfies (1.1). Suppose that $M$ is Ricci-parallel, that is, $\nabla Q = 0$. From (1.1) and (2.3) we have
\[Q\xi = 2n\kappa\xi. \tag{4.1}\]

From (2.4) and (4.1), we have
\[\langle \nabla ZQ \rangle \xi = -2n\kappa(\phi + \phi h)Z + Q(\phi + \phi h)Z. \tag{4.2}\]

Since $M$ is Ricci-parallel, we have
\[Q(\phi + \phi h)Z = 2n\kappa(\phi + \phi h)Z \tag{4.3}\]
for any vector field $Z$ on $M$. If we substitute $Z$ with $\phi Z$, then by using (2.1) and (4.1), we obtain that
\[Q(I - h) = 2n\kappa(I - h). \tag{4.4}\]

If $\kappa = 1 (h \equiv 0)$, then from (4.4) we see that $M$ is Einstein-Sasakian and the scalar curvature $\tau = 2n(2n + 1)$.

Now, we assume that $\kappa \neq 1$, that is, $M$ is non-Sasakian. Differentiating (2.14) covariantly, then it follows that
\[(\nabla ZQ)X = [2(n - 1) + \mu] (\nabla Z h)X - [2(1 - n) + n(2\kappa + \mu)] g((\phi + \phi h)Z, X)\xi - [2(1 - n) + n(2\kappa + \mu)] \eta(X)(\phi + \phi h)Z, \tag{4.5}\]
and thus we get
\[[2(n - 1) + \mu](\nabla Z h)X = [2(1 - n) + n(2\kappa + \mu)] [g((\phi + \phi h)Z, X)\xi + \eta(X)(\phi + \phi h)Z]. \tag{4.6}\]

Together with (2.12) we have
\[[2(n - 1) + \mu] [(1 - \kappa) \{2(1 - n) + n(2\kappa + \mu)\} g(Z, \phi X) + g(Z, h\phi X)] \xi + \eta(X)(h\phi + h\phi h)Z - \mu \eta(Z)\phi hX \]
\[= [2(1 - n) + n(2\kappa + \mu)] [g((\phi + \phi h)Z, X)\xi + \eta(X)(\phi + \phi h)Z]. \tag{4.7}\]

If we put $Z = \xi$ in (4.7), then we have
\[\mu [2(n - 1) + \mu] \phi h = 0, \tag{4.8}\]
and hence we see that \( \mu = 0 \) or \( 2(n - 1) + \mu = 0 \). Now, we discuss our arguments divided into two cases: (i) \( \mu = 0 \), (ii) \( 2(n - 1) + \mu = 0 \).

The case (i) \( \mu = 0 \). Then (4.7) becomes

\[
2(n - 1)[(1 - \kappa)\{1 - \kappa\}g(Z, \phi X) + g(Z, h \phi X)]\xi + \eta(X)(h \phi + h \phi h)Z = [2(1 - n) + 2n\kappa]\{g((\phi + \phi h)Z, X)\xi + \eta(X)(\phi + \phi h)Z\}. \tag{4.9}
\]

Putting \( X = \xi \), then by using (2.2) and (2.3) we get

\[
2(1 - n)(\phi h + h^2)Z = [2(1 - n) + 2n\kappa] (\phi + \phi h)Z. \tag{4.10}
\]

We apply \( \phi \) and use (2.2), then we have

\[
2(n - 1)h^2Z + 2n khZ + [2(1 - n) + 2n\kappa](Z - \eta(Z)\xi) = 0. \tag{4.11}
\]

Since the trace of \( h^2 = 2n(1 - \kappa) \) and the trace of \( h = 0 \), we have \( \kappa = 0 \). Thus, \( M \) satisfies \( R(X, Y)\xi = 0 \). By Theorem 2.1 we conclude that \( M \) is locally the product of \((n + 1)\)-dimensional manifold and an \( n \)-dimensional manifold of positive constant curvature 4.

The case (ii) \( 2(n - 1) + \mu = 0 \). Then (2.14) is reduced to

\[
Q = [2(n - 1) - n\mu]I + [2(1 - n) + n(2\kappa + \mu)]\eta \otimes \xi, \tag{4.12}
\]

that is, \( M \) is \( \eta \)-Einstein. From (4.7) we get

\[
[2(1 - n) + n(2\kappa + \mu)]\{g(-(\phi + \phi h)Z, X)\xi + \eta(X)(\phi + \phi h)Z\} = 0 \tag{4.13}
\]

for any vector field \( X, Z \) on \( M \). Putting \( X = \xi \) in (4.13), then we have

\[
[2(1 - n) + n(2\kappa + \mu)](\phi + \phi h)Z = 0. \tag{4.14}
\]

If \( 2(1 - n) + n(2\kappa + \mu) \neq 0 \), since \( \mu = 2(1 - n) \) we have

\[
\kappa = \frac{n^2 - 1}{n}. \tag{4.15}
\]

But we know that \( \kappa < 1 \), and thus we see that \( n \) must be equal to 1 and hence \( \kappa = \mu = 0 \). Otherwise, \( 2(1 - n) + n(2\kappa + \mu) = 0 \), then (4.14) becomes

\[
\phi + \phi h = 0, \tag{4.16}
\]

which is impossible. Therefore, summing up all the arguments in this section we have Theorem 1.2.

\textbf{Remark 4.1.} \( \mathbb{R}^3(x^1, x^2, x^3) \) or \( T^3 \) (torus) with \( \eta = 1/2(\cos x^3 dx^1 + \sin x^3 dx^2) \) and \( g_{ij} = 1/4\delta_{ij} \) is an \( \eta \)-Einstein, non-Sasakian, contact Riemannian manifold (cf. [1]).
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