A THEOREM OF MEIR-KEELER TYPE REVISITED

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(Received 1 October 1998)

ABSTRACT. In 1993, the authors presented a fixed point theorem of Meir-Keeler type. The proposed proof of a lemma—on which the said theorem depends on—is invalid. In this note, we alter the statement of this lemma and give a valid proof thereof, so that the main result of the previous paper is still true.

Keywords and phrases. Compatible maps of type (A), a generalized (ϵ, δ) - $\{S, T\}$ -contraction, common fixed point.

2000 Mathematics Subject Classification. Primary 54H25, 47H10.

In 1993, we introduced the concept of compatible maps of type (A) and “proved” the following theorem.

THEOREM 1 [1, Theorem 3.2]. *Let A, B, S and T be mappings of a complete metric space (X, d) . Suppose that the pair $\{A, B\}$ is a generalized (ϵ, δ) - $\{S, T\}$ -contraction with δ lower semi-continuous. If the following conditions are satisfied:*

- (i) *one of A, B, S or T is continuous, and*
 - (ii) *the pairs A, S and B, T are compatible of type (A) on X ,*
- then A, B, S and T have a unique common fixed point in X .*

The purpose of this note is to ensure that the above is indeed true. This is necessary since the proof of Theorem 1 relies on Lemma 3.1 in [1]. However, the proof of part (1) of this lemma is faulty and the proof of part (2) is not “tight.” In the following, we provide a thorough and complete proof of Lemma 4 below which is a “reshuffled and revamped” version of Lemma 3.1 in [1]. This accomplishes our mission, since the proof of Theorem 1 is valid if the lemma is true.

The proof of part (3) of Lemma 4 below is much like the proof of Lemma 3.1(c) in [2] with minor initial modifications. We include all the proof of part (3) for ease of reading and completeness sake. We need the following definitions given in [1].

DEFINITION 2 [2]. Let A, B, S and T be mappings of a metric space (X, d) into itself such that $A(X) \subset T(X)$ and $B(X) \subset S(X)$. For $x_0 \in X$, any sequence $\{y_n\}$ defined by

$$\begin{aligned} y_{2n-1} &= Tx_{2n-1} = Ax_{2n-2}, \\ y_{2n} &= Sx_{2n} = Bx_{2n-1} \end{aligned} \tag{1}$$

for $n \in \mathbb{N}$ (the set of positive integers) is called an $\{S, T\}$ -iteration of x_0 under A and B .

The following definition was given in [1], but erroneously required that $\delta(\epsilon) < \epsilon$.

DEFINITION 3. Let A, B, S and T be mappings of a metric space (X, d) into itself. The pair $\{A, B\}$ is called a *generalized (ϵ, δ) - $\{S, T\}$ -contraction* if

$$A(X) \subset T(X), \quad B(X) \subset S(X) \quad (2)$$

and there exists a function $\delta : (0, \infty) \rightarrow (0, \infty)$ such that, for any $\epsilon > 0, \delta(\epsilon) > \epsilon$, and

$$\begin{aligned} \epsilon \leq M(x, y) = \max \{ & d(Sx, Ty), d(Sx, Ax), d(Ty, By), \\ & \frac{1}{2}(d(Sx, By) + d(Ty, Ax)) \} < \delta(\epsilon) \Rightarrow d(Ax, By) < \epsilon. \end{aligned} \quad (3)$$

Now we state and prove a modified version of the lemma in question.

LEMMA 4. Let A, B, S and T be mappings of a metric space (X, d) into itself and let the pair $\{A, B\}$ be a generalized (ϵ, δ) - $\{S, T\}$ -contraction. If $x_0 \in X$ and $\{y_n\}$ is an $\{S, T\}$ iteration of x_0 under A and B , we have the following:

(1) $\lim_{n \rightarrow \infty} d(y_n, y_{n+1}) = 0$.

(2) For every $\epsilon > 0$, there exists $n_1 \in \mathbb{N}$ such that, whenever $p, q \geq n_1$ and of opposite parity,

$$\epsilon \leq d(y_p, y_q) < \epsilon + r \Rightarrow d(y_{p+1}, y_{q+1}) < \epsilon, \quad (4)$$

where $r = \min\{\epsilon/2, (\delta(\epsilon) - \epsilon)/2\}$.

(3) The sequence $\{y_n\}$ is a Cauchy sequence in X .

PROOF. To prove part (1), first note that, by (3),

$$\begin{aligned} d(Ax, By) &= 0, \quad \text{if } M(x, y) = 0, \\ d(Ax, By) &< M(x, y), \quad \text{otherwise.} \end{aligned} \quad (5)$$

Thus, $d(Ax, By) \leq M(x, y)$ for $x, y \in X$. Therefore, if $x_0 \in X$, (1) and (3) imply that

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &= d(Bx_{2n-1}, Ax_{2n}) \\ &= d(Ax_{2n}, Bx_{2n-1}) \leq M(x_{2n}, x_{2n-1}) \\ &= \max \{ d(Sx_{2n}, Tx_{2n-1}), d(Sx_{2n}, Ax_{2n}), d(Tx_{2n-1}, Bx_{2n-1}), \\ &\quad \frac{1}{2}(d(Sx_{2n}, Bx_{2n-1}) + d(Tx_{2n-1}, Ax_{2n})) \} \\ &= \max \{ d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n}), \\ &\quad \frac{1}{2}(d(y_{2n}, y_{2n}) + d(y_{2n-1}, y_{2n+1})) \} \\ &\leq \max \{ d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n+1}), \\ &\quad \frac{1}{2}(d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})) \} \\ &\leq \max \{ d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n+1}) \}. \end{aligned} \quad (6)$$

Now if $M(x_{2n}, x_{2n-1}) = 0$, by the above, we know

$$d(y_{2n}, y_{2n-1}) = d(y_{2n}, y_{2n+1}) = 0. \quad (7)$$

But if $M(x_{2n}, x_{2n-1}) > 0$, (5) and the above imply that

$$d(y_{2n}, y_{2n+1}) < M(x_{2n}, x_{2n-1}) \leq \max \{d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n+1})\}, \quad (8)$$

i.e.,

$$d(y_{2n}, y_{2n+1}) < d(y_{2n}, y_{2n-1}). \quad (9)$$

Thus, in any event, we have

$$d(y_{2n}, y_{2n+1}) \leq M(x_{2n}, x_{2n-1}) \leq d(y_{2n}, y_{2n-1}) \quad (10)$$

for $n \in X$. Similarly,

$$d(y_{2n+1}, y_{2n+2}) \leq M(x_{2n+1}, x_{2n}) \leq d(y_{2n+1}, y_{2n}) \quad (11)$$

for $n \in X$. Thus $s = \{d(y_k, y_{k+1})\}$ is nonincreasing and is bounded below by 0. Hence, s converges to $t \in [0, \infty)$, the greatest lower bound of s . If $t = 0$, we are done. So, suppose that $t > 0$. Since s converges in a nonincreasing manner to t , (10) yields $m \in \mathbb{N}$ such that

$$t \leq M(x_{2m}, x_{2m-1}) < \delta(t). \quad (12)$$

But then (3) implies that

$$d(Ax_{2m}, Bx_{2m-1}) = d(y_{2m+1}, y_{2m}) < t, \quad (13)$$

which contradicts the fact that t is the greatest lower bound of s . Thus part (1) is true.

Now we prove part (2). Let $\epsilon > 0$. Part (1) permits us to choose $n_1 \in \mathbb{N}$ such that

$$d(y_n, y_{n+1}) < \frac{r}{2} \quad \text{for } n \geq n_1, \quad (14)$$

where $r = \min\{\epsilon/2, (\delta(\epsilon) - \epsilon)/2\}$. Let $p, q \in \mathbb{N}$ such that $p, q \geq n_1$, where $p = 2n$ and $q = 2m - 1$. Suppose that

$$\epsilon \leq d(y_p, y_q) = d(y_{2n}, y_{2m-1}) < \epsilon + r. \quad (15)$$

Keeping (1), (3), (14) and (15) in mind, we can write the following:

$$\begin{aligned} \epsilon &\leq d(y_p, y_q) = d(Sx_{2n}, Tx_{2m-1}) \leq M(x_{2n}, x_{2m-1}) \\ &= \max \left\{ d(Sx_{2n}, Tx_{2m-1}), d(Sx_{2n}, Ax_{2n}), d(Tx_{2m-1}, Bx_{2m-1}), \right. \\ &\quad \left. \frac{1}{2}(d(Sx_{2n}, Bx_{2m-1}) + d(Tx_{2m-1}, Ax_{2n})) \right\} \\ &= \max \left\{ d(y_p, y_q), d(y_p, y_{p+1}), d(y_q, y_{q+1}), \right. \\ &\quad \left. \frac{1}{2}(d(y_p, y_{q+1}), d(y_q, y_{p+1})) \right\} \\ &= \max \left\{ d(y_p, y_q), \frac{1}{2}(d(y_p, y_{q+1}) + d(y_q, y_{p+1})) \right\} \\ &\leq \max \left\{ d(y_p, y_q), \frac{1}{2}(2d(y_p, y_q) + d(y_q, y_{q+1}) + d(y_p, y_{p+1})) \right\} \\ &\leq d(y_p, y_q) + \frac{r}{2} < \epsilon + \frac{3r}{2} < \epsilon + 2r \leq \epsilon + (\delta(\epsilon) - \epsilon) = \delta(\epsilon). \end{aligned} \quad (16)$$

Thus we have $\epsilon \leq M(x_{2n}, x_{2m-1}) < \delta(\epsilon)$, and so (3) implies that

$$d(y_{p+1}, y_{q+1}) = d(Ax_{2n}, bx_{2m-1}) < \epsilon, \quad (17)$$

as desired.

To prove part (3), let $\alpha = 2\epsilon > 0$ and let $r = \min\{\epsilon/2, (\delta(\epsilon) - \epsilon)/2\}$. Part (2) of the lemma yields $n_1 \in \mathbb{N}$ such that, whenever $p, q \in \mathbb{N}$ and $p, q > n_1$, then

$$d(y_{p+1}, y_{q+1}) < \epsilon \text{ if } \epsilon \leq d(y_p, y_q) < \epsilon + r \text{ and } p, q \text{ are of opposite parity.} \quad (18)$$

And part (1) of the lemma permits us to choose $n_0 \in \mathbb{N}$ such that $n_0 > n_1$ and

$$d(y_m, y_{m+1}) < \frac{r}{6} \quad (19)$$

for $m \geq n_0$. Now we let $q > p \geq n_0$ —so that both (18) and (19) hold—and show that $d(y_p, y_q) < \alpha$, thereby proving that $\{y_n\}$ is a Cauchy sequence in X . So suppose that

$$d(y_p, y_q) \geq \alpha = 2\epsilon. \quad (20)$$

To show that (20) produces a contradiction, we first want to choose an $m > p$ such that

$$\epsilon + \frac{r}{3} < d(y_p, y_m) < \epsilon + r \text{ with } p \text{ and } m \text{ of opposite parity.} \quad (21)$$

To this end, let k be the smallest integer greater than p such that $d(y_p, y_k) > \epsilon + (r/2)$. The integer k exists by (20) since $r < \epsilon$. Moreover, we have

$$d(y_p, y_k) < \epsilon + \frac{2r}{3}. \quad (22)$$

For otherwise, $\epsilon + 2r/3 \leq d(y_p, y_{k-1}) + d(y_{k-1}, y_k) < d(y_p, y_{k-1}) + r/6$, since $k-1 \geq p \geq n_0 > n_1$, and therefore

$$\epsilon + \frac{r}{2} < d(y_p, y_{k-1}). \quad (23)$$

Since $k-1 \geq p$, (23) implies that $k-1 > p$. But then (23) contradicts the choice of k . We thus have

$$\epsilon + \frac{r}{2} < d(y_p, y_k) < \epsilon + \frac{2r}{3}. \quad (24)$$

So, if p and k are of opposite parity, we can let $m = k$ in (24) to obtain (21). If p and k are of like parity, p and $k+1$ are opposite parity. Since $d(y_k, y_{k+1}) < r/6$ by (19), the triangle inequality and (24) imply that

$$\epsilon + \frac{r}{3} < d(y_p, y_{k+1}) < \epsilon + \frac{5r}{6}. \quad (25)$$

In this instance, we let $m = k+1$. In any event, by (24) and (25), we can choose m such that m and p are of opposite parity and (21) holds. But then, since $p, m \geq n_0$, (19) and (21) imply that

$$\epsilon + \frac{r}{3} < d(y_p, y_m) \leq d(y_p, y_{p+1}) + d(y_{p+1}, y_{m+1}) + d(y_{m+1}, y_m). \quad (26)$$

Therefore, by (21) and (18), we have

$$\epsilon + \frac{r}{3} < \frac{r}{3} + d(y_{p+1}, y_{m+1}) < \frac{r}{3} + \epsilon. \quad (27)$$

This is the anticipated contradiction. This completes the proof. $\square$

ACKNOWLEDGEMENTS. The authors wish to thank Dr. A. S. Ranadive and Dr. A. K. Sinha for alerting them to the difficulties in [1]. The first author (YJC) was supported in part by the Basic Science Research Institute Program, Ministry of Education, Korea, 1998, Project No. BSRI-98-1405.

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