NONLINEAR VARIATIONAL EVOLUTION INEQUALITIES IN HILBERT SPACES

JIN-MUN JEONG, DOO-HOAN JEONG, and JONG-YEOUL PARK

(Received 31 July 1998)

ABSTRACT. The regular problem for solutions of the nonlinear functional differential equations with a nonlinear hemicontinuous and coercive operator A and a nonlinear term $f(\cdot, \cdot): x'(t) + Ax(t) + \partial\phi(x(t)) \ni f(t, x(t)) + h(t)$ is studied. The existence, uniqueness, and a variation of solutions of the equation are given.

Keywords and phrases. Nonlinear variational evolution inequality, maximal monotone operator, subdifferential operator, regularity.

2000 Mathematics Subject Classification. Primary 34G20; Secondary 35F25.

1. Introduction. Let H and V be two real separable Hilbert spaces such that V is a dense subspace of H . Let the operator A be given a single-valued operator, which is hemicontinuous and coercive from V to V^* . Here V^* stands for the dual space of V . Let $\phi: V \rightarrow (-\infty, +\infty]$ be a lower semicontinuous, proper convex function. Then the subdifferential operator $\partial\phi: V \rightarrow V^*$ of ϕ is defined by

$$\partial\phi(x) = \{x^* \in V^*; \phi(x) \leq \phi(y) + (x^*, x - y), y \in V\}, \quad (1.1)$$

where $(\cdot, \cdot)$ denotes the duality pairing between V^* and V . We are interested in the following nonlinear functional differential equation on H :

$$\begin{aligned} \frac{dx(t)}{dt} + Ax(t) + \partial\phi(x(t)) &\ni f(t, x(t)) + h(t), \quad 0 < t \leq T, \\ x(0) &= x_0, \end{aligned} \quad (1.2)$$

where the nonlinear mapping f is a Lipschitz continuous from $\mathbb{R} \times V$ into H . Equation (1.2) is caused by the following nonlinear variational inequality problem:

$$\begin{aligned} \left(\frac{dx(t)}{dt} + Ax(t), x(t) - z \right) + \phi(x(t)) - \phi(z) \\ \leq (f(t, x(t)) + h(t), x(t) - z), \quad \text{a.e., } 0 < t \leq T, z \in V, \\ x(0) = x_0. \end{aligned} \quad (1.3)$$

If A is a linear continuous symmetric operator from V into V^* and satisfies the coercive condition, then equation (1.2), which is called the linear parabolic variational inequality, is extensively studied in Barbu [5, Sec. 4.3.2] (also see [4, Sec. 4.3.1]). The existence of solutions for the semilinear equation with similar conditions for nonlinear term f have been dealt with in [1, 2, 6]. Using more general hypotheses for

nonlinear term $f(\cdot, x)$, we intend to investigate the existence and the norm estimate of a solution of the above nonlinear equation on $L^2(0, T; V) \cap W^{1,2}(0, T; V^*)$, which is also applicable to optimal control problem. A typical example was given in the last section.

2. Perturbation of subdifferential operator. Let H and V be two real Hilbert spaces. Assume that V is a dense subspace in H and the injection of V into H is continuous. If H is identified with its dual space we may write $V \subset H \subset V^*$ densely and the corresponding injections are continuous. The norm on V (respectively H) will be denoted by $\|\cdot\|$ (respectively $|\cdot|$). The duality pairing between the element v_1 of V^* and the element v_2 of V is denoted by (v_1, v_2) , which is the ordinary inner product in H if $v_1, v_2 \in H$. For the sake of simplicity, we may consider

$$\|u\| \leq |u| \leq \|u\|_*, \quad u \in V, \quad (2.1)$$

where $\|\cdot\|_*$ is the norm of the element of V^* .

REMARK 2.1. If an operator A_0 is bounded linear from V to V^* and generates an analytic semigroup, then it is easily seen that

$$H = \left\{ x \in V^* : \int_0^T \|A_0 e^{tA_0} x\|_*^2 dt < \infty \right\} \quad \text{for the time } T > 0. \quad (2.2)$$

Therefore, in terms of the intermediate theory we can see that

$$(V, V^*)_{1/2,2} = H, \quad (2.3)$$

where $(V, V^*)_{1/2,2}$ denotes the real interpolation space between V and V^* .

We note that nonlinear operator A is said to be hemicontinuous on V if

$$\text{w-lim}_{t \rightarrow 0} A(x + ty) = Ax \quad \text{for every } x, y \in V, \quad (2.4)$$

where “w-lim” indicates the weak convergence on V . Let $A : V \rightarrow V^*$ be given a single valued and hemicontinuous from V to V^* such that

$$\begin{aligned} A(0) = 0, \quad (Au - Av, u - v) &\geq \omega_1 \|u - v\|^2 - \omega_2 |u - v|^2, \\ \|Au\|_* &\leq \omega_3 (\|u\| + 1) \end{aligned} \quad (2.5)$$

for every $u, v \in V$, where $\omega_2 \in \mathcal{R}$ and ω_1, ω_3 are some positive constants. Here, we note that if $A(0) \neq 0$ we need the following assumption:

$$(Au, u) \geq \omega_1 \|u\|^2 - \omega_2 |u|^2 \quad \text{for every } u \in V. \quad (2.6)$$

It is also known that $A + \omega_2 I$ is maximal monotone and $R(A + \omega_2 I) = V^*$ where $R(A + \omega_2 I)$ is the range of $A + \omega_2 I$ and I is the identity operator.

First, let us be concerned with the following perturbation of subdifferential operator:

$$\frac{dx(t)}{dt} + Ax(t) + \partial\phi(x(t)) \ni h(t), \quad 0 < t \leq T, \quad x(0) = x_0. \quad (2.7)$$

To prove the regularity for the nonlinear equation (1.2) without the nonlinear term $f(\cdot, x)$ we apply the method in [5, Sec. 4.3.2].

PROPOSITION 2.1. *Let $h \in L^2(0, T; V^*)$ and $x_0 \in V$ satisfying that $\phi(x_0) < \infty$. Then (2.7) has a unique solution*

$$x \in L^2(0, T; V) \cap C([0, T]; H) \quad (2.8)$$

which satisfies

$$\|x\|_{L^2 \cap C} \leq C_1 (1 + \|x_0\| + \|h\|_{L^2(0, T; V^*)}), \quad (2.9)$$

where C_1 is a constant and $L^2 \cap C = L^2(0, T; V) \cap C([0, T]; H)$.

PROOF. Substituting $v(t) = e^{\omega_2 t} x(t)$ we can rewrite (2.7) as follows:

$$\begin{aligned} \frac{dv(t)}{dt} + (A + \omega_2 I)v(t) + e^{-\omega_2 t} \partial \phi(v(t)) &\ni e^{-\omega_2 t} h(t), \quad 0 < t \leq T, \\ v(0) &= e^{\omega_2 t} x_0. \end{aligned} \quad (2.10)$$

Then the regular problem for (2.7) is equivalent to that for (2.10). Consider the operator $L : D(L) \subset H \rightarrow H$

$$\begin{aligned} Lv &= \{Av + e^{-\omega_2 t} \partial \phi(v) + \omega_2 v\} \cap H \quad \forall v \in D(L), \\ D(L) &= \{v \in V; \{Av + e^{-\omega_2 t} \partial \phi(v) + \omega_2 v\} \cap H \neq \emptyset\}. \end{aligned} \quad (2.11)$$

Since $A + \omega_2 I$ is a monotone, hemicontinuous and bounded operator from V into V^* and $e^{-\omega_2 t} \partial \phi$ is maximal monotone, we infer in [4, Cor. 1.1 of Ch. 2] that L is maximal monotone. Then in [5, Thm. 1.4] (also see [4, Thm. 2.3, Cor. 2.1]), for every $x_0 \in D(L)$ and $h \in W^{1,1}([0, T]; H)$, the Cauchy problem (2.10) has a unique solution $v \in W^{1,\infty}([0, T]; H)$. Let us assume that $x_0 \in D(L)$ and $h \in W^{1,2}(0, T; H)$. Multiplying (2.7) by $x - x_0$ and using (2.5) and the maximal monotonicity of $\partial \phi$ it holds

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |x(t) - x_0|^2 + \omega_1 |x(t) - x_0|^2 \\ \leq \omega_2 |x(t) - x_0| + (h(t) - Ax_0 - \partial \phi(x_0), x(t) - x_0). \end{aligned} \quad (2.12)$$

Since

$$\begin{aligned} (h(t) - Ax_0 - \partial \phi(x_0), x(t) - x_0) &\leq \|h(t) - Ax_0 - \partial \phi(x_0)\|_* \|x(t) - x_0\| \\ &\leq \frac{1}{2c} \|h(t) - Ax_0 - \partial \phi(x_0)\|_*^2 + \frac{c}{2} \|x(t) - x_0\|^2 \end{aligned} \quad (2.13)$$

for every real number c , so using Gronwall's inequality, the inequality (2.12) implies that

$$|x(t) - x_0|^2 + \int_0^t \|x(s) - x_0\|^2 ds \leq C_1 \left(\int_0^t \|h(s)\|_*^2 ds + \|x_0\|^2 + 1 \right) \quad (2.14)$$

for some positive constant C_1 , that is,

$$\|x\|_{L^2(0, T; V) \cap C([0, T]; H)} \leq C_1 (1 + \|x_0\| + \|h\|_{L^2(0, T; V^*)}). \quad (2.15)$$

Hence we have proved (2.9). Let $x_0 \in V$ such that $\phi(x_0) < \infty$ and $h \in L^2(0, T; V^*)$. Then there exist sequences $\{x_{0n}\} \subset D(L)$ and $\{h_n\} \subset W^{1,2}(0, T; H)$ such that $x_{0n} \rightarrow x_0$

in V and $h_n \rightarrow h$ in $L^2(0, T; V^*)$ as $n \rightarrow \infty$. Let $x_n \in W^{1, \infty}(0, T; H)$ be the solution of (2.7) with initial value x_{0n} and with h_n instead of h . Since $\partial\phi$ is monotone, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} |x_n(t) - x_m(t)|^2 + \omega_1 \|x_n(t) - x_m(t)\|^2 \\ & < (h_n(t) - h_m(t), x_n(t) - x_m(t)) + \omega_2 |x_n(t) - x_m(t)|^2 \\ & \leq \frac{1}{2c} \|h_n(t) - h_m(t)\|_*^2 + \frac{c}{2} \|x_n(t) - x_m(t)\|^2 \\ & \quad + \omega_2 |x_n(t) - x_m(t)|^2, \quad \text{a.e., } t \in (0, T) \end{aligned} \quad (2.16)$$

for every real number c . Therefore, if we choose $\omega_1 - (c/2)$ then by integrating over $[0, T]$ and using Gronwall's inequality it follows that

$$\begin{aligned} & |x_n(t) - x_m(t)| + 2 \left(\omega_1 - \frac{c}{2} \right) \|x_n(t) - x_m(t)\|_{L^2(0, T; V)} \\ & \leq e^{2\omega_2 T_1} (|x_{0n} - x_{0m}| + c^{-1} \|h_n - h_m\|_{L^2(0, T; V^*)}), \end{aligned} \quad (2.17)$$

and hence, we have that $\lim_{n \rightarrow \infty} x_n(t) = x(t)$ exists in H . Furthermore, x satisfies (2.7). Indeed, for all $0 \leq s < t \leq T$ and $y \in \partial\phi(x)$, multiplying (2.7) by $x(t) - x$ and integrating over $[s, t]$ we have

$$\begin{aligned} & \frac{1}{2} (|x(t) - x|^2 - |x(s) - x|^2) \leq \int_s^t (h(\tau) - Ax - y, x(\tau) - x) d\tau \\ & \quad + \omega_2 \int_s^t |x(\tau) - x|^2 d\tau, \end{aligned} \quad (2.18)$$

and, therefore,

$$\begin{aligned} & \left(\frac{x(t) - x(s)}{t - s}, x(s) - x \right) \leq \frac{1}{t - s} \int_s^t (h(\tau) - Ax - y, x(\tau) - x) d\tau \\ & \quad + \frac{\omega_2}{t - s} \int_s^t |x(\tau) - x|^2 d\tau. \end{aligned} \quad (2.19)$$

This implies

$$\left(\frac{d}{dt} x(t), x(t) - x \right) \leq (h(t) - Ax - y + \omega_2(x(t) - x), x(t) - x), \quad (2.20)$$

a.e., $t \in (0, T)$, that is,

$$\left(\frac{d}{dt} x(t) - h(t) - \omega_2 x(t) + (Ax + y + \omega_2 x), x(t) - x \right) \leq 0. \quad (2.21)$$

Since $A + \partial\phi + \omega_2 I$ is maximal monotone, we have

$$\frac{d}{dt} x(t) - h(t) - \omega_2 x(t) \in -(A + \partial\phi + \omega_2 I)x(t), \quad \text{a.e., } t \in (0, T). \quad (2.22)$$

Thus, the proof is complete. $\square$

COROLLARY 2.1. Assume the hypotheses as in Proposition 2.1, in addition, assume that $\partial\phi$ satisfies the growth condition as follows:

$$\|z\|_* \leq M(|x| + 1), \quad \text{a.e., } x \in D(\phi), z \in \partial\phi(x). \quad (2.23)$$

Then equation (2.7) has a unique solution

$$x \in L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \cap C([0, T]; H) \quad (2.24)$$

which satisfies

$$\|x\|_{L^2 \cap W^{1,2} \cap C} \leq C(1 + \|x_0\| + \|h\|_{L^2(0, T; V^*)}). \quad (2.25)$$

PROOF. From (2.7) and (2.23) it follows that

$$\left\| \frac{d}{dt} x(t) \right\|_* + \omega_1 \|x(t)\| \leq \omega_2 |x(t)| + M(|x(t)| + 1) + \|h(t)\|_*. \quad (2.26)$$

Hence, by virtue of (2.15) we have that

$$\|x\|_{W^{1,2}(0, T; H)} \leq C_2(1 + \|x_0\| + \|h\|_{L^2(0, T; V^*)}). \quad (2.27)$$

□

REMARK 2.2. If V is compactly imbedded in H , the imbedding $L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \subset L^2(0, T; H)$ is compact in Aubin [3, Rem. 1, Thm. 2]. Hence, the mapping $h \mapsto x$ is compact from $L^2(0, T; H)$ to $L^2(0, T; H)$.

3. Nonlinear integrodifferential equation. Let $f : [0, T] \times V \rightarrow H$ be a nonlinear mapping satisfying the following variational evolution inequality:

$$|f(t, x) - f(t, y)| \leq L\|x - y\|, \quad f(t, 0) = 0 \quad (3.1)$$

for a positive constant L .

THEOREM 3.1. Let (2.5) and (3.1) be satisfied. Then (1.2) has a unique solution

$$x \in L^2(0, T; V) \cap C([0, T]; H). \quad (3.2)$$

Furthermore, there exists a constant C_2 such that

$$\|x\|_{L^2 \cap C} \leq C_2(1 + \|x_0\| + \|h\|_{L^2(0, T; V^*)}). \quad (3.3)$$

If $(x_0, h) \in V \times L^2(0, T; V^*)$, then $x \in L^2(0, T; V) \cap C([0, T]; H)$ and the mapping

$$V \times L^2(0, T; V^*) \ni (x_0, h) \mapsto x \in L^2(0, T; V) \cap C([0, T]; H) \quad (3.4)$$

is continuous.

PROOF. Let $y \in L^2(0, T; V)$. Then from (3.1), $f(\cdot, y(\cdot)) \in L^2(0, T; H)$. Thus, by virtue of Proposition 2.1 we know that the problem

$$\begin{aligned} \frac{dx(t)}{dt} + Ax(t) + \partial\phi(x(t)) &\ni f(t, y(t)) + h(t), \quad 0 < t \leq T, \\ x(0) &= x_0 \end{aligned} \quad (3.5)$$

has a unique solution $x_y \in L^2(0, T; V) \cap C([0, T]; H)$, where x_y is the solution of (3.5).

Let us choose a constant $c > 0$ such that

$$\omega_1 - \frac{c}{2} > 0, \quad (3.6)$$

and let us fix $T_0 > 0$ so that

$$(2c\omega_1 - c^2)^{-1} e^{2\omega_2 T_0} L < 1. \quad (3.7)$$

Let $x_i, i = 1, 2$, be the solution of (3.5) corresponding to y_i . Then, by the monotonicity of $\partial\phi$, it follows that

$$\begin{aligned} & (\dot{x}_1(t) - \dot{x}_2(t), x_1(t) - x_2(t)) + (Ax_1(t) - Ax_2(t), x_1(t) - x_2(t)) \\ & \leq (f(t, y_1(t)) - f(t, y_2(t)), x_1(t) - x_2(t)), \end{aligned} \quad (3.8)$$

and hence, using the assumption (2.5), we have that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} |x_1(t) - x_2(t)|^2 + \omega_1 \|x_1(t) - x_2(t)\|^2 \\ & \leq \omega_2 |x_1(t) - x_2(t)|^2 + \|f(t, y_1(t)) - f(t, y_2(t))\|_* \|x_1(t) - x_2(t)\|. \end{aligned} \quad (3.9)$$

Since

$$\begin{aligned} & \|f(t, y_1(t)) - f(t, y_2(t))\|_* \|x_1(t) - x_2(t)\| \\ & \leq \frac{1}{2c} \|f(t, y_1(t)) - f(t, y_2(t))\|_*^2 + \frac{c}{2} \|x_1(t) - x_2(t)\|^2 \end{aligned} \quad (3.10)$$

for every $c > 0$ and by integrating (3.9) over $(0, T_0)$ we have

$$\begin{aligned} & |x_1(T_0) - x_2(T_0)|^2 + (2\omega_1 - c) \int_0^{T_0} \|x_1(t) - x_2(t)\|^2 dt \\ & \leq \frac{1}{c} \|f(t, y_1) - f(t, y_2)\|_{L^2(0, T_0; V^*)}^2 + 2\omega_2 \int_0^{T_0} |x_1(t) - x_2(t)|^2 dt, \end{aligned} \quad (3.11)$$

and by Gronwall's inequality,

$$\|x_1 - x_2\|_{L^2(0, T_0; V)}^2 \leq (2c\omega_1 - c^2)^{-1} e^{2\omega_2 T_0} \|f(t, y_1) - f(t, y_2)\|_{L^2(0, T_0; V^*)}^2. \quad (3.12)$$

Thus, from (3.1) it follows that

$$\|x_1 - x_2\|_{L^2} \leq (2c\omega_1 - c^2)^{-1} e^{2\omega_2 T_0} L \|y_1 - y_2\|_{L^2(0, T_0; V)}. \quad (3.13)$$

Hence we have proved that $y \mapsto x$ is a strictly contraction from $L^2(0, T_0; V)$ to itself if condition (3.7) is satisfied. It shows that (1.2) has a unique solution in $[0, T_0]$.

Let y be the solution of

$$\frac{dy(t)}{dt} + Ay(t) + \partial\phi(y(t)) \ni 0, \quad 0 < t \leq T_0, \quad y(0) = x_0. \quad (3.14)$$

Then, since

$$\frac{d}{dt} (x(t) - y(t)) + (Ax(t) - Ay(t)) + (\partial\phi(x(t)) - \partial\phi(y(t))) \ni f(t, x(t)) + h(t), \quad (3.15)$$

multiplying by $x(t) - y(t)$ and using the monotonicity of $\partial\phi$, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} |x(t) - y(t)|^2 + \omega_1 \|x(t) - y(t)\|^2 \\ & \leq \omega_2 |x(t) - y(t)|^2 + \|f(t, x(t)) + h(t)\|_* \|x(t) - y(t)\|. \end{aligned} \quad (3.16)$$

Therefore, putting

$$N = (2c\omega_1 - c^2)^{-1} e^{2\omega_2 T_0}, \quad (3.17)$$

from (3.1), it follows that

$$\begin{aligned} \|\mathbf{x} - \mathbf{y}\|_{L^2(0, T_0; V)} &\leq N \|f(\cdot, \mathbf{x}) + \mathbf{h}\|_{L^2(0, T_0; V^*)} \\ &\leq NL \|\mathbf{x}\|_{L^2(0, T_0; V)} + N \|\mathbf{h}\|_{L^2(0, T_0; V^*)}, \end{aligned} \quad (3.18)$$

and hence

$$\begin{aligned} \|\mathbf{x}\|_{L^2(0, T_0; V)} &\leq \frac{1}{1 - NL} \|\mathbf{y}\|_{L^2(0, T_0; V)} + N \|\mathbf{h}\|_{L^2(0, T_0; V^*)} \\ &\leq \frac{C_1}{1 - NL} \left(1 + \|\mathbf{x}_0\| + N \|\mathbf{h}\|_{L^2(0, T_0; V^*)}\right) \\ &\leq C_2 \left(1 + \|\mathbf{x}_0\| + \|\mathbf{h}\|_{L^2(0, T_0; V^*)}\right) \end{aligned} \quad (3.19)$$

for some positive constant C_2 . Since condition (3.7) is independent of the initial values, the solution of (1.2) can be extended to the interval $[0, nT_0]$ for natural number n , i.e., for the initial value $\mathbf{x}(nT_0)$ in the interval $[nT_0, (n+1)T_0]$, as analogous estimate (3.19) holds for the solution in $[0, (n+1)T_0]$. Furthermore, similar to (2.12) and (2.15) in Section 2, the estimate (3.3) is easily obtained.

Now we prove the last result. If $(\mathbf{x}_0, \mathbf{h}) \in V \times L^2(0, T; V^*)$ then $\mathbf{x}$ belongs to $L^2(0, T; V)$. Let $(\mathbf{x}_{0i}, \mathbf{h}_i) \in V \times L^2(0, T; V^*)$ and $\mathbf{x}_i$ be the solution of (1.2) with $(\mathbf{x}_{0i}, \mathbf{h}_i)$ in place of $(\mathbf{x}_0, \mathbf{h})$ for $i = 1, 2$. Multiplying (1.2) by $\mathbf{x}_1(t) - \mathbf{x}_2(t)$, we have

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|\mathbf{x}_1(t) - \mathbf{x}_2(t)\|^2 + \omega_1 \|\mathbf{x}_1(t) - \mathbf{x}_2(t)\|^2 \\ &\leq \omega_2 \|\mathbf{x}_1(t) - \mathbf{x}_2(t)\|^2 + \|f(t, \mathbf{x}_1(t)) - f(t, \mathbf{x}_2(t))\|_* \|\mathbf{x}_1(t) - \mathbf{x}_2(t)\| \\ &\quad + \|\mathbf{h}_1(t) - \mathbf{h}_2(t)\|_* \|\mathbf{x}_1(t) - \mathbf{x}_2(t)\|. \end{aligned} \quad (3.20)$$

If $\omega_1 - c/2 > 0$, we can choose a constant $c_1 > 0$ so that

$$\begin{aligned} &\omega_1 - \frac{c}{2} - \frac{c_1}{2} > 0, \\ &\|\mathbf{h}_1(t) - \mathbf{h}_2(t)\|_* \|\mathbf{x}_1(t) - \mathbf{x}_2(t)\| \leq \frac{1}{2c_1} \|\mathbf{h}_1(t) - \mathbf{h}_2(t)\|_*^2 + \frac{c_1}{2} \|\mathbf{x}_1(t) - \mathbf{x}_2(t)\|^2. \end{aligned} \quad (3.21)$$

Let $T_1 < T$ be such that

$$2\omega_1 - c - c_1 - c^{-1} e^{2\omega_2 T_1} L > 0. \quad (3.22)$$

Integrating (3.20) over $[0, T_1]$, where $T_1 < T$ and as seen in the first part of the proof, it follows that

$$\begin{aligned}
& (2\omega_1 - c - c_1) \|x_1 - x_2\|_{L^2(0, T_0; V)}^2 \\
& \leq e^{2\omega_2 T_1} \left\{ \|x_{01} - x_{02}\| + \frac{1}{c} \|f(t, x_1) - f(t, x_2)\|_{L^2(0, T_0; V^*)}^2 + \frac{1}{c_1} \|h_1 - h_2\|_{L^2(0, T_0; V^*)} \right\} \\
& \leq e^{2\omega_2 T_1} \left\{ \|x_{01} - x_{02}\| + \frac{1}{c} L \|x_1 - x_2\|_{L^2(0, T_0; V)}^2 + \frac{1}{c_1} \|h_1 - h_2\|_{L^2(0, T_0; V^*)} \right\}.
\end{aligned} \tag{3.23}$$

Putting

$$N_1 = 2\omega_1 - c - c_1 - c^{-1} e^{2\omega_2 T_1} L, \tag{3.24}$$

we have

$$\|x_1 - x_2\|_{L^2} \leq \frac{e^{2\omega_2 T_1}}{N_1} \left(\|x_{01} - x_{02}\| + \frac{1}{c_1} \|h_1 - h_2\| \right). \tag{3.25}$$

Suppose $(x_{0n}, h_n) \rightarrow (x_0, h)$ in $V \times L^2(0, T; V^*)$, and let x_n and x be the solutions of (1.2) with (x_{0n}, h_n) and (x_0, h) , respectively. Then, by virtue of (3.25) and (3.20), we see that $x_n \rightarrow x$ in $L^2(0, T_1, V) \cap C([0, T_1]; H)$. This implies that $x_n(T_1) \rightarrow x(T_1)$ in V . Therefore the same argument shows that $x_n \rightarrow x$ in

$$L^2(T_1, \min\{2T_1, T\}; V) \cap C([T_1, \min\{2T_1, T\}]; H). \tag{3.26}$$

Repeating this process, we conclude that $x_n \rightarrow x$ in $L^2(0, T; V) \cap C([0, T]; H)$.

If $\partial\phi$ satisfies the growth condition (2.23) as is seen in Corollary 2.1, we can obtain the following result. $\square$

COROLLARY 3.1. *Let (2.5), (3.1), and the growth condition (2.23) be satisfied. Then (1.2) has a unique solution*

$$x \in L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \subset C([0, T]; H). \tag{3.27}$$

Furthermore, there exists a constant C_2 such that

$$\|x\|_{L^2(0, T; V) \cap W^{1,2}(0, T; V^*)} \leq C_2 \left(1 + \|x_0\| + \|h\|_{L^2(0, T; V^*)} \right). \tag{3.28}$$

If $(x_0, h) \in V \times L^2(0, T; V^*)$, then $x \in L^2(0, T; V) \cap W^{1,2}(0, T; V^*)$ and the mapping

$$V \times L^2(0, T; V^*) \ni (x_0, h) \mapsto x \in L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \tag{3.29}$$

is continuous.

EXAMPLE. Let Ω be a region in an n -dimensional Euclidean space $\mathbb{R}^n$ with boundary $\partial\Omega$ and closure $\overline{\Omega}$. For an integer $m \geq 0$, $C^m(\Omega)$ is the set of all m -times continuously differentiable functions in Ω , and $C_0^m(\Omega)$ is its subspace consisting of functions with compact supports in Ω . If $m \geq 0$ is an integer and $1 \leq p \leq \infty$, $W^{m,p}(\Omega)$ is the set of all functions f whose derivative $D^\alpha f$ up to degree m in the distribution sense belong to $L^p(\Omega)$. As usual, the norm of $W^{m,p}(\Omega)$ is given by

$$\|f\|_{m,p} = \left(\sum_{|\alpha| \leq m} \|D^\alpha f\|_p^p \right)^{1/p} = \left\{ \sum_{|\alpha| \leq m} \int_\Omega |D^\alpha f(x)|^p dx \right\}^{1/p}, \tag{3.30}$$

where $1 \leq p < \infty$ and $D^0 f = f$. In particular, $W^{0,p}(\Omega) = L^p(\Omega)$ with the norm $\|\cdot\|_{0,p}$.

$W_0^{m,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$. For $p' = p/(p-1)$, $1 < p < \infty$, $W^{-m,p}(\Omega)$ stands the dual space $W_0^{m,p'}(\Omega)$ of $W_0^{m,p}(\Omega)$ whose norm is denoted by $\|\cdot\|_{-m,p}$.

We take $V = W_0^{m,2}(\Omega)$, $H = L^2(\Omega)$ and $V^* = W^{-m,2}(\Omega)$ and consider a nonlinear differential operator of the form

$$Ax = \sum_{|\alpha| \leq m} (-D)^\alpha A_\alpha(u, x, \dots, D^m x), \quad (3.31)$$

where $A_\alpha(u, \xi)$ are real functions defined on $\Omega \times \mathbb{R}^N$ and satisfy the following conditions:

(1) A_α are measurable in u and continuous in ξ . There exists $k \in L^2(\Omega)$ and a positive constant C such that

$$A_\alpha(u, 0) = 0, \quad |A_\alpha(u, \xi)| \leq C(|\xi| + k(u)), \quad \text{a.e., } u \in \Omega, \quad (3.32)$$

where $\xi = (\xi_\alpha; |\alpha| \leq m)$.

(2) For every $(\xi, \eta) \in \mathbb{R}^N \times \mathbb{R}^N$ and for almost every $u \in \Omega$ the following condition holds:

$$\sum_{|\alpha| \leq m} (A_\alpha(u, \xi) - A_\alpha(u, \eta))(\xi_\alpha - \eta_\alpha) \geq \omega_1 \|\xi - \eta\|_{m,2} - \omega_2 \|\xi - \eta\|_{0,2}, \quad (3.33)$$

where $\omega_2 \in \mathbb{R}$ and ω_1 is a positive constant.

Let the sesquilinear form $a : V \times V \rightarrow \mathbb{R}$ be defined by

$$a(x, y) = \sum_{|\alpha| \leq m} \int_{\Omega} A_\alpha(u, x, \dots, D^m x) D^\alpha y \, du. \quad (3.34)$$

Then by Lax-Milgram theorem we know that the associated operator $A : V \rightarrow V^*$, defined by

$$(Ax, y) = a(x, y), \quad x, y \in V, \quad (3.35)$$

is monotone and semicontinuous and satisfies conditions (2.5) in Section 2.

Let $g(t, u, x, p)$, $p \in \mathbb{R}^m$, be assumed that there is a continuous $\rho(t, r) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^+$ and a real constant $1 \leq \gamma$ such that

$$\begin{aligned} g(t, u, 0, 0) &= 0, \\ |g(t, u, x, p) - g(t, u, x, q)| &\leq \rho(t, |x|)(1 + |p|^{\gamma-1} + |q|^{\gamma-1})|p - q|, \\ |g(t, u, x, p) - g(t, u, y, p)| &\leq \rho(t, |x| + |y|)(1 + |p|^\gamma)|x - y|. \end{aligned} \quad (3.36)$$

Let

$$f(t, x)(u) = g(t, u, x, Dx, D^2 x, \dots, D^m x). \quad (3.37)$$

Then noting that

$$\begin{aligned} \|f(t, x) - f(t, y)\|_{0,2}^2 &\leq 2 \int_{\Omega} |g(t, u, x, p) - g(t, u, x, q)|^2 \, du \\ &\quad + 2 \int_{\Omega} |g(t, u, x, q) - g(t, u, y, q)|^2 \, du, \end{aligned} \quad (3.38)$$

where $p = (Dx, \dots, D^m x)$ and $q = (Dy, \dots, D^m y)$, it follows from (3.36) that

$$\|f(t, x) - f(t, y)\|_{0,2}^2 \leq L(\|x\|_{m,2}, \|y\|_{m,2}) \|x - y\|_{m,2}, \quad (3.39)$$

where $L(\|x\|_{m,2}, \|y\|_{m,2})$ is a constant depending on $\|x\|_{m,2}$ and $\|y\|_{m,2}$.

Let $\phi: V \rightarrow (-\infty, +\infty]$ be a lower semicontinuous, proper convex function. Then for $x_0 \in W_0^{m,2}(\Omega)$ satisfying that $\phi(x_0) < \infty$ and $h \in L^2(0, T; W^{-m,2}(\Omega))$, (1.2) is caused by the following nonlinear variational inequality problem:

$$\begin{aligned} & \left(\frac{dx(t)}{dt} + Ax(t), x(t) - z \right) + \phi(x(t)) - \phi(z) \\ & \leq (f(t, x(t)) + h(t), x(t) - z), \quad a.e., 0 < t \leq T, z \in W_0^{m,2}(\Omega), \\ & x(0) = x_0 \end{aligned} \quad (3.40)$$

has a unique solution

$$x \in L^2(0, T; W_0^{m,2}(\Omega)) \cap C([0, T]; L^2(\Omega)). \quad (3.41)$$

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JEONG: DIVISION OF MATHEMATICAL SCIENCES, PUKYONG NATIONAL UNIVERSITY, PUSAN 608-737, KOREA

E-mail address: jmjeong@pknu.dolphin.ac.kr

JEONG: DONGEUI TECHNICAL JUNIOR COLLEGE, PUSAN 614-053, KOREA

PARK: DEPARTMENT OF MATHEMATICS, PUSAN NATIONAL UNIVERSITY, PUSAN 609-739, KOREA

E-mail address: jyepark@hyowon.pusan.ac.kr