SPATIAL NUMERICAL RANGES OF ELEMENTS OF SUBALGEBRAS OF $C_0(X)$

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To Professor Junzo Wada on his retirement from Waseda University

ABSTRACT. When $A$ is a subalgebra of the commutative Banach algebra $C_0(X)$ of all continuous complex-valued functions on a locally compact Hausdorff space $X$, the spatial numerical range of element of $A$ can be described in terms of positive measures.

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1. Introduction and results. Let $X$ be a locally compact Hausdorff space and $C_0(X)$ the commutative Banach algebra (with supremum norm $\|\|_\infty$) of all continuous complex-valued functions on $X$ which vanish at infinity. Let $A$ be a subalgebra (not necessarily closed) of $C_0(X)$, $A^*$ the dual space of $A$ and $f \in A$. If $A$ is unital, then

$$V_1(A,f) \equiv \{ m(f) : m \in A^* and \|m\|=m(1)=1 \} \quad (1.1)$$

is called the (algebra) numerical range of $f$ and it is a nonempty compact convex subset of the complex plane $C$ (cf. [1, page 52]). However if $A$ is nonunital, then the above definition is not meaningful. In this case, Gaur and Husain [2] introduced the following set:

$$V(A,f) = \{ m(fg) : \exists m \in A^* and g \in A \text{ such that } \|m\| = \|g\|_\infty = m(g) = 1 \} \quad (1.2)$$

and studied the spatial numerical range in a nonunital algebra. The set $V(A,f)$ is equal to $V_1(A,f)$ whenever $A$ is unital.

In [2], Gaur and Husain proved the following result.

**Theorem 1.1.** Let $f$ be an element of $C_0(X)$. Then

$$\text{co} \, R(f) \subseteq V(C_0(X),f) \subseteq \overline{\text{co}} \, R(f), \quad (1.3)$$

where $\text{co}$ and $\overline{\text{co}}$ denote the convex hull and the closed hull, respectively, and $R(f)$ is the range of the function $f$.

In this paper, we describe spatial numerical ranges of elements of subalgebras of $C_0(X)$ in terms of positive measures and show that Theorem 1.1 also holds for the subalgebra of $C_0(X)$. Let $M(X)$ denote the measure space of all bounded regular Borel measures on $X$. Our main result is the following theorem.
**Theorem 1.2.** Let $A$ be a subalgebra of $C_0(X)$ and $f \in A$. Then

(i) $V(A,f) = \{ \int f d|\mu| : \exists \mu \in M(X) \text{ and } \exists g \in A \text{ such that } \|\mu\| = \|g\|_{\infty} = \int g d\mu = 1 \} \subseteq \overline{\text{co} R(f)}$, where $|\mu|$ denotes the total variation of $\mu$.

(ii) If $A$ has the following property (#), then $\text{co} R(f) \subseteq V(A,f)$.

(#) For any finite set $\{x_1, \ldots, x_n\}$ in $X$, there exists $g \in A$ such that $\|g\|_{\infty} = 1$ and $g(x_1) = \cdots = g(x_n) = 1$.

**Corollary 1.3.** Let $A$ be a $^*$-subalgebra of $C_0(X)$ and $f \in A$. Then

(i) \[ V(A,f) = \left\{ \int f d\mu : \exists \mu \in M(X) \text{ and } \exists g \in A \text{ such that } \|\mu\| = 1, \mu \geq 0, 0 \leq g \leq 1 \text{ and } \int g d\mu = 1 \right\}. \] (1.4)

(ii) If $A$ has the following property (##), then \[ V(A,f) = \left\{ \int f d\mu : 0 \leq \mu \in M(X), \|\mu\| = 1 \text{ and } \text{supp} (\mu) \text{ is compact} \right\}. \] (1.5)

(##) For any compact set $E \subseteq X$, there exists $g \in A$ such that $0 \leq g \leq 1$ and $g(x) = 1$ for all $x \in E$. Here $\text{supp}(\mu)$ denotes the support of $\mu$.

**Remark 1.4.** If $A = C_0(X)$, then $A$ satisfies the desired properties appeared in Theorem 1.2 and Corollary 1.3. Hence, we have

\[ V(C_0(X), f) = \left\{ \int f d\mu : 0 \leq \mu \in M(X), \|\mu\| = 1 \text{ and } \text{supp}(\mu) \text{ is compact} \right\}. \] (1.6)

and

\[ \text{co} R(f) \subseteq V(C_0(X), f) \subseteq \overline{\text{co} R(f)}. \] (1.7)

2. Proofs of results

**Proof of Theorem 1.2.** (i) By the Hahn-Banach extension theorem, we have

\[ V(A,f) = \left\{ \int f g d\mu : \exists \mu \in M(X) \text{ and } \exists g \in A \text{ such that } \|\mu\| = \|g\|_{\infty} = \int g d\mu = 1 \right\}. \] (2.1)

Now suppose $\mu \in M(X)$, $g \in A$ and $\|\mu\| = \|g\|_{\infty} = \int g d\mu = 1$. Let $\mu = h \cdot |\mu|$ be the polar decomposition of $\mu$ (cf. [4, Corollary 19.38]), then $|h| = 1$. Since

\[ 1 = \int g d\mu = \int g h d|\mu| \leq \sqrt{\int |g|^2 d|\mu|} \sqrt{\int |h|^2 d|\mu|} \leq \|\mu\| = 1, \] (2.2)

it follows that

\[ \left| \int g h d|\mu| \right| \leq \sqrt{\int |g|^2 d|\mu|} \sqrt{\int |h|^2 d|\mu|} \] (2.3)
and hence there exists a scalar \( \lambda \in \mathbb{C} \) such that \( \frac{\overline{g(x)}}{h(x)} |\mu| \) a.e. on \( X \). Therefore we have

\[
1 = \int g h d|\mu| = \overline{\lambda} \int |h|^2 d|\mu| = \overline{\lambda} \|\mu\| = \overline{\lambda},
\]

and so

\[
\int f g d\mu = \int f g h d|\mu| = \int f |h|^2 d|\mu| = \int f d|\mu|.
\]

Consequently, we obtain that

\[
V(A, f) = \left\{ \int f d|\mu| : \exists \mu \in M(X) \text{ and } \exists g \in A \text{ such that } \|\mu\| = \|g\|_{\infty} = \int g d\mu = 1 \right\}.
\]

Next consider the following set:

\[
S = \left\{ v \in M(X) : v \geq 0 \text{ and } \|v\| = \int |g|^2 dv = 1 \right\}.
\]

Then \( S \) is a weak \( \ast \)-closed set. Also note that \( \sqrt{\int |g|^2 d|\mu|} = 1 \) by the above arguments and hence \( |\mu| \in S \). Moreover, we can easily see that any extreme point of \( S \) is also an extreme point of \( \left\{ v \in M(X) : v \geq 0, \|v\| \leq 1 \right\} \). But since the extreme points of \( \left\{ v \in M(X) : v \geq 0, \|v\| \leq 1 \right\} \) consist of \( 0 \) and \( \{ \delta_x : x \in X \} \), it follows that the extreme points of \( S \) are contained in \( \{ \delta_x : x \in X \} \), where \( \delta_x \) denotes the Dirac measure at \( x \in X \). Then by the Krein-Milman theorem, we have \( S \subseteq \overline{\text{co}} \{ \delta_x : x \in X \} \) and so \( |\mu| \in \overline{\text{co}} R(f) \). Therefore, we have

\[
\left\{ \int f d|\mu| : \exists \mu \in M(X) \text{ and } \exists g \in A \text{ such that } \|\mu\| = \|g\|_{\infty} = \int g d\mu = 1 \right\} \subseteq \overline{\text{co}} R(f).
\]

(ii) Let \( x_1, \ldots, x_n \in X \) and \( \lambda_1 \geq 0, \ldots, \lambda_n \geq 0 \) with \( \lambda_1 + \cdots + \lambda_n = 1 \), and set \( \mu = \lambda_1 \delta_{x_1} + \cdots + \lambda_n \delta_{x_n} \). Then \( \mu \) is a positive measure on \( X \) with norm of one such that \( \int f d\mu = \lambda_1 f(x_1) + \cdots + \lambda_n f(x_n) \). Assume that \( A \) has the property (\#). Then we can take an element \( g \in A \) such that \( \|g\|_{\infty} = 1 \) and \( g(x_1) = \cdots = g(x_n) = 1 \). Therefore \( \int g d\mu = 1 \) and hence we conclude that \( \text{co} R(f) \subseteq V(A, f) \). \( \square \)

**Proof of Corollary 1.3.** Assume \( A \) is a \( \ast \)-subalgebra of \( C_0(X) \).

(i) Set

\[
W = \left\{ \int f d\mu : \exists \mu \in M(X) \text{ and } \exists g \in A \text{ such that } \|\mu\| = 1, \mu \geq 0, 0 \leq g \leq 1 \text{ and } \int g d\mu = 1 \right\}.
\]

Then \( W \subseteq V(A, f) \) by Theorem 1.2. Now, suppose \( \mu \in M(X) \), \( g \in A \) and \( \|\mu\| = \|g\|_{\infty} = \int g d\mu = 1 \). Then \( \|\mu\| = 1 \) and \( 0 \leq |g|^2 \leq 1 \). Also since \( A \) is a \( \ast \)-subalgebra
of $C_0(X)$, we have $|g|^2 = g\overline{g} \in A$. Moreover, we have $\int |g|^2 d|\mu| = 1$ and hence $\int |g|^2 d|\mu| = 1$ as observed in the proof of Theorem 1.2. Hence we conclude that $V(A, f) \subseteq W$ by Theorem 1.2 again and hence $V(A, f) = W$.

(ii) Let $\mu \in M(X)$, $g \in A$ and $||\mu|| = ||g||_\infty = \int g d\mu = 1$. Then we have $\int (1 - |g|^2) d|\mu| = 0$ as observed in the proof of (i). It follows that $|g(x)| = 1$ $|\mu|$ a.e. on $X$ and hence supp$(||\mu||)$ is compact. Therefore, we have

$$V(A, f) = \left\{ \int f d\mu : 0 \leq \mu \in M(X), \ ||\mu|| = 1 \text{ and supp}(\mu) \text{ is compact} \right\}. \quad (2.10)$$

Now, suppose that $0 \leq v \in M(X)$, $||v|| = 1$ and supp$(v)$ is compact and that $A$ has the property ($##$). Then we can take an element $g \in A$ such that $0 \leq g \leq 1$ and $g(x) = 1$ for all $x \in \text{supp}(v)$. Therefore $||v|| = ||g|| = \int g dv = 1$ and hence, by Theorem 1.2, we have

$$\left\{ \int f d\mu : 0 \leq \mu \in M(X), \ ||\mu|| = 1 \text{ and supp}(\mu) \text{ is compact} \right\} \subseteq V(A, f). \quad (2.11)$$

3. Examples. Let $X = (0, 1]$, the half open interval and let $h \in C_0(X)$ be such that $h(x) \neq 0$ for all $x \in X$. Define

$$A = \{ hg : g \in C_0(X) \}. \quad (3.1)$$

Then $A$ is an ideal (and hence subalgebra) of $C_0(X)$. In this case, $A$ is neither closed nor unital. Also $A$ has the desired property: for any compact set $E \subseteq X$, there exists $g \in A$ such that $||g||_\infty = 1$ and $g(x) = 1$ for all $x \in E$. In fact, let $t_E = \min\{x : x \in E\}$ and so $0 < t_E \leq 1$. Put

$$g_0(x) = \begin{cases} \frac{xp(x)}{t_E \varphi(t_E)h(x)}, & \text{if } 0 < x \leq t_E, \\ \frac{1}{h(x)}, & \text{if } t_E < x \leq 1, \end{cases} \quad (3.2)$$

where $\varphi(x) = \min\{|h(t_E)|x/t_E, |h(x)|\} \ (x \in X)$. Since $|g_0(x)| \leq x/t_E \varphi(t_E)$ for $0 < x \leq t_E$, a function $g_0$ must be in $C_0(X)$. Set $g = h g_0$ and hence $g \in A$ and $g(x) = 1$ for all $x \in E$. Also since $|g(x)| = (x/t_E) \cdot (\varphi(x)/\varphi(t_E)) \leq 1 \cdot 1 = 1$ for $0 < x \leq t_E$, it follows that $||g||_\infty = 1$. Therefore $A$ has the desired property and so by Theorem 1.2, we have

$$V(A, f) = \left\{ \int f d|\mu| : \exists \mu \in M(X) \text{ and } g \in A \\
\text{such that } ||\mu|| = ||g||_\infty = \int g d|\mu| = 1 \right\}. \quad (3.3)$$

and

$$\text{co} R(f) \subseteq V(A, f) \subseteq \overline{\text{co}} R(f) \quad (3.4)$$

for every $f \in A$. In particular, if $f \in A$ is real-valued, then we have

$$V(A, f) = \left\{ \begin{array}{ll} [\alpha, \beta], & \text{if } \{x \in X : f(x) = 0\} \neq \emptyset, \\ (0, \beta) \text{ or } [\alpha, 0), & \text{if } \{x \in X : f(x) = 0\} = \emptyset, \end{array} \right. \quad (3.5)$$

where $\alpha = \inf\{f(x) : x \in X\}$ and $\beta = \sup\{f(x) : x \in X\}$.
Of course, this holds even if \( A = C_0(X) \), so we have the spatial numerical range of the function \( f(x) = x(x \in X) \) with respect to \( C_0(X) \) is equal to \( X = (0,1] \). This fact has been observed in [2, Example 4.2].

Also, \( A \) is not generally a \( * \)-subalgebra of \( C_0(X) \). But if \( h \) is real-valued, then \( A \) becomes a \( * \)-subalgebra of \( C_0(X) \) and so \( A \) has the property \((\#)\).

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**References**


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