RANDOM TRILINEAR FORMS AND THE SCHUR MULTIPLICATION OF TENSORS

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ABSTRACT. We obtain estimates for the distribution of the norm of the random trilinear form $A : \ell_r^M \times \ell_p^N \times \ell_q^K \rightarrow \mathbb{C}$, defined by $A(e_i, e_j, e_k) = a_{ijk}$, where the a_{ijk} 's are uniformly bounded, independent, mean zero random variables. As an application, we make progress on the problem when $\ell_r \otimes \ell_p \otimes \ell_q$ is a Banach algebra under the Schur multiplication.

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1. Introduction and notation. We adopt the standard notation ℓ_p^N ($1 \leq p < \infty$) for the complex vector space $\mathbb{C}^N$ equipped with the norm

$$\|x\|_p := \left(\sum_{n=1}^N |x_n|^p \right)^{1/p}. \quad (1.1)$$

The usual modifications are made to define ℓ_∞^N and the infinite dimensional sequence spaces ℓ_p ($1 \leq p \leq \infty$). All of these are Banach spaces.

Let $A : \ell_p^N \rightarrow \ell_q^M$ ($1 \leq p, q \leq \infty$) be a linear map and define the operator norm by

$$\|A\|_{p \rightarrow q} := \sup \{ \|Ax\|_q : \|x\|_p = 1 \}. \quad (1.2)$$

The map A can be represented as an $M \times N$ matrix (a_{ij}) with respect to the standard bases. Motivated by problems on absolutely summing operators, Bennett [1] and Bennett, Goodman and Newman [2] obtained estimates for the probability distribution of $\|A\|_{p \rightarrow q}$ when the a_{ij} 's are independent, mean zero random variables bounded by (2.3). They showed that, for all $1 \leq p, q \leq \infty$, the expectation $\mathcal{E}(\|A\|_{p \rightarrow q})$ is of the same order as the smallest possible value of $\|A\|_{p \rightarrow q}$ when all the matrix entries a_{ij} are ± 1 .

Notice that these results can also be interpreted as estimates for the norms of random bilinear forms. Problems involving the von Neumann inequality led Varopoulos [9] to work with norms of random trilinear forms on ℓ_2^N . His results were extended and refined by Mantero and Tonge [7]. Let $A : \ell_{p_1}^N \times \cdots \times \ell_{p_n}^N \rightarrow \mathbb{C}$ be an n -linear form with $A(e_{k_1}, \dots, e_{k_n}) = a_{k_1, \dots, k_n}$, where the e_k 's are the standard unit basis vectors. There is a natural norm

$$\|A\|_{p_1, \dots, p_n} := \sup \{ |A(x_1, \dots, x_n)| : \|x_i\|_{p_i} \leq 1 (1 \leq i \leq n) \}. \quad (1.3)$$

In [7], it was shown that when the $a_{k_1, \dots, k_n}$'s are independent random variables taking the values ± 1 with equal probability, the expectation $\mathcal{E}(\|A\|_{p_1, \dots, p_n})$ is of the same

order as the smallest conceivable value: the least possible value of $\|A\|_{p_1, \dots, p_n}$ when each $a_{k_1 \dots k_n}$ is ± 1 . These results turned out to be useful in the study of Banach algebra structures on the tensor products $\ell_{p_1} \otimes \dots \otimes \ell_{p_n}$. However, open problems were left, even in the case $n = 3$.

We address these problems. They involve the Schur product of tensors $A, B \in \ell_r \otimes \ell_p \otimes \ell_q$ when $A = (a_{ijk})$ and $B = (b_{ijk})$ this is given by

$$A * B := (a_{ijk} b_{ijk}). \quad (1.4)$$

The injective tensor product $\ell_r \tilde{\otimes} \ell_p \tilde{\otimes} \ell_q$ is the completion of $\ell_r \otimes \ell_p \otimes \ell_q$ under the norm

$$\|(a_{ijk})\|_{rpq} := \sup \left\{ \left\| \sum_{ijk} a_{ijk} x_i y_j z_k \right\| : \|x\|_{r'} \leq 1, \|y\|_{p'} \leq 1, \|z\|_{q'} \leq 1 \right\}. \quad (1.5)$$

Here, the index p' is the one conjugate to p , that is $1/p + 1/p' = 1$. Notice that $\|A\|_{rpq} = \|A\|_{r', p', q'}$.

In [7], it was shown that the Schur product extends continuously to $\ell_r \tilde{\otimes} \ell_p \tilde{\otimes} \ell_q$ when

- (i) the sum of the reciprocals of any two of p, q , and r is at least $3/2$,
- (ii) $1 \leq p, q, r \leq 2$ and the sum of the reciprocals of all three is at least 2 , or
- (iii) at least one of p, q , and r is 1 or ∞ .

Cases were also identified where the Schur product did not extend continuously to $\ell_r \tilde{\otimes} \ell_p \tilde{\otimes} \ell_q$. We extend knowledge of such cases by providing an estimate for the distribution of $\|A\|_{rpq}$ when the a_{ijk} 's are uniformly bounded, independent, mean zero random variables. Our methods build on the techniques of [1, 2, 7].

Although the problem we consider is relevant to many issues in the geometry of Banach spaces or Banach algebras (see, for example, Diestel, Jarchow, and Tonge [4]), we are not aware of any progress in the last few years. Recent work on the Schur product (see, for example, Horn and Johnson [6]) or on random matrices (see, for example, Girko [5]) mostly focuses on other issues. There is one notable exception, namely, the body of work on completely bounded operators. The basic theory can be found in Paulsen [8], and interesting results closely related to the operator algebra theory, developed by Varopoulos [9] and his group appear in Blecher and Le Merdy [3] and references therein. None of this, however, appears to be directly applicable to the problem we treat in this paper.

2. The probabilistic estimate. We consistently use $\mathcal{P}$ to denote probability and $\mathbb{E}$ to denote mathematical expectation.

PROPOSITION 2.1. *Let $1 \leq p \leq 2$ and $2 \leq q, r < \infty$. Let $A = (a_{ijk}) \in \ell_r^M \tilde{\otimes} \ell_p^N \tilde{\otimes} \ell_q^K$ and suppose that the a_{ijk} 's are independent, mean zero random variables, and that each $|a_{ijk}| \leq 1$. Then there are positive constants C_1 and C_2 , independent of M, N , and K , such that*

$$\mathcal{P} \left(\|A\|_{rpq}^r \geq C_1 M N^{(r/p)-(r/2)} + C_2 (N+K) N^{(r/p)-1} K^{(r/q)-(2/q)} \right) < 1. \quad (2.1)$$

PROOF. Our argument is an adaptation of the work in [1, 2]. Note that

$$\|A\|_{r p q}^r = \sup \left\{ \sum_{i=1}^M \left| \sum_{j,k=1}^{N,K} a_{ijk} \mathcal{Y}_j z_k \right|^r : \|\mathcal{Y}\|_{p'} \leq 1, \|z\|_{q'} \leq 1 \right\}. \quad (2.2)$$

As in [1] or [2], for any positive λ and any nonzero $(\mathcal{Y}_j)_{j=1}^N$ and $(z_k)_{k=1}^K$, we have

$$\mathcal{P} \left(\left| \sum_{j,k=1}^{N,K} a_{ijk} \mathcal{Y}_j z_k \right| \geq \lambda \right) \leq 2 \exp \left(-\frac{\lambda^2}{4} \sum_{j,k=1}^{N,K} \mathcal{Y}_j^2 z_k^2 \right). \quad (2.3)$$

If $\mu > 0$, then

$$\begin{aligned} \mathcal{E} \left(\exp \left(\mu \left| \sum_{j,k=1}^{N,K} a_{ijk} \mathcal{Y}_j z_k \right|^r \right) \right) &= \int_0^\infty e^{\mu \lambda^r} d\mathcal{P} \left(\left| \sum_{j,k=1}^{N,K} a_{ijk} \mathcal{Y}_j z_k \right| \leq \lambda \right) \\ &= 1 + \int_0^\infty \mu r \lambda^{r-1} e^{\mu \lambda^r} \mathcal{P} \left(\left| \sum_{j,k=1}^{N,K} a_{ijk} \mathcal{Y}_j z_k \right| > \lambda \right) d\lambda, \end{aligned} \quad (2.4)$$

and an application of (2.3) gives

$$\mathcal{E} \left(\exp \left(\mu \left| \sum_{j,k=1}^{N,K} a_{ijk} \mathcal{Y}_j z_k \right|^r \right) \right) \leq 1 + \int_0^\infty \mu r \lambda^{r-1} e^{\mu \lambda^r} \cdot 2e^{-\lambda^2/4 \sum_{j,k=1}^{N,K} \mathcal{Y}_j^2 z_k^2} d\lambda. \quad (2.5)$$

Since $2 \leq r < \infty$, if $0 \leq \mu \leq (N^{1/p} K^{1/q})^{2-r} / 8 \sum_{j,k=1}^{N,K} \mathcal{Y}_j^2 z_k^2$, we can find a constant C_1 , independent of M, N , or K , such that

$$\begin{aligned} \mathcal{E} \left(\exp \left(\mu \left| \sum_{j,k=1}^{N,K} a_{ijk} \mathcal{Y}_j z_k \right|^r \right) \right) &\leq 1 + 2\mu r \int_0^\infty \lambda^{r-1} e^{-\lambda^2/8 \sum_{j,k=1}^{N,K} \mathcal{Y}_j^2 z_k^2} d\lambda \\ &= 1 + C_1 \mu \left(\sum_{j,k=1}^{N,K} \mathcal{Y}_j^2 z_k^2 \right)^{r/2}. \end{aligned} \quad (2.6)$$

Next, applying independence, we obtain

$$\begin{aligned} \mathcal{E} \left(\exp \left(\mu \sum_{i=1}^M \left| \sum_{j,k=1}^{N,K} a_{ijk} \mathcal{Y}_j z_k \right|^r \right) \right) &= \prod_{i=1}^M \mathcal{E} \left(\exp \left(\mu \left| \sum_{j,k=1}^{N,K} a_{ijk} \mathcal{Y}_j z_k \right|^r \right) \right) \\ &\leq \prod_{i=1}^M \left(1 + C_1 \mu \left(\sum_{j,k=1}^{N,K} \mathcal{Y}_j^2 z_k^2 \right)^{r/2} \right) \\ &\leq \exp \left(C_1 M \mu \left(\sum_{j,k=1}^{N,K} \mathcal{Y}_j^2 z_k^2 \right)^{r/2} \right). \end{aligned} \quad (2.7)$$

Consequently, for any $\nu > 0$, we have

$$\mathcal{P} \left(\mu \sum_{i=1}^M \left| \sum_{j,k=1}^{N,K} a_{ijk} \mathcal{Y}_j z_k \right|^r \geq C_1 M \mu \left(\sum_{j,k=1}^{N,K} \mathcal{Y}_j^2 z_k^2 \right)^{r/2} + \nu \right) \leq e^{-\nu}. \quad (2.8)$$

Now, if $\|\mathcal{Y}\|_{p'} \leq 1$ and $\|z\|_{q'} \leq 1$, then since $1 \leq p \leq 2 \leq q < \infty$, we have $\sum_{j,k=1}^{N,K} \mathcal{Y}_j^2 z_k^2 \leq N^{(2/p)-1}$. Using the result of the entropy argument in the proof [7, Thm. 1], we get

$$\mathbb{P} \left(\|a\|_{rpq}^r \geq C_1 M N^{(r/p)-(r/2)} + \frac{\nu}{\mu} \right) \leq e^{D(N+K)} e^{-\nu/2^{(3r+1)}}, \quad (2.9)$$

where D is some positive constant independent of M , N , and K .

Take $\mu = 1/8(N^{1/p}K^{1/q})^{2-r}N^{-(2/p)+1}$ and $\nu = 2^{3r+2}D(N+K)$, and set $C_2 = 2^{3r+5}D$ in (2.9) to get

$$\mathbb{P} \left(\|A\|_{rpq}^r \geq C_1 M N^{(r/p)-(r/2)} + C_2 (N+K) (N^{1/p}K^{1/q})^{r-2} N^{(2/p)-1} \right) \leq e^{-D(N+K)}. \quad (2.10)$$

Since $e^{-D(N+K)} < 1$ for large N and K , the result follows. $\square$

What we need later is an immediate corollary.

COROLLARY 2.2. *Let $1 \leq p \leq 2$ and $2 \leq q, r < \infty$. Then there is an $A = (a_{ijk}) \in \ell_r^M \tilde{\otimes} \ell_p^N \tilde{\otimes} \ell_q^K$, with each $a_{ijk} = \pm 1$, such that*

$$\|A\|_{rpq}^2 < C \max \left(M^{2/r} N^{(2/p)-1}, N^{2/p} K^{(2/q)(1-2/r)}, N^{(2/p)-(2/r)} K^{(2/r)+(2/q)(1-2/r)} \right), \quad (2.11)$$

where C is a positive constant independent of M , N , and K .

The next proposition and its corollary are obtained by making minor adjustments to the arguments above. We present them without proof.

PROPOSITION 2.3. *Let $2 \leq p, q, r < \infty$. Let $A = (a_{ijk}) \in \ell_r^M \tilde{\otimes} \ell_p^N \tilde{\otimes} \ell_q^K$ and suppose that the a_{ijk} 's are independent, mean zero random variables, and that each $|a_{ijk}| \leq 1$. Then there are positive constants C_1 and C_2 , independent of M , N and K , such that*

$$\mathbb{P} \left(\|A\|_{rpq}^r \geq C_1 M + C_2 (N+K) N^{(r/p)-(2/p)} K^{(r/q)-(2/q)} \right) < 1. \quad (2.12)$$

COROLLARY 2.4. *Let $2 \leq p, q, r < \infty$. Then there is an $A = (a_{ijk}) \in \ell_r^M \tilde{\otimes} \ell_p^N \tilde{\otimes} \ell_q^K$, with each $a_{ijk} = \pm 1$, such that*

$$\|A\|_{rpq}^2 < C \max \left(M^{2/r}, N^{(2/r)+(2/p)(1-2/r)}, K^{(2/q)(1-2/r)}, N^{(2/p)(1-2/r)} K^{(2/r)+(2/q)(1-2/r)} \right), \quad (2.13)$$

where C is a positive constant independent of M , N , and K .

3. Application to the question of the continuity of Schur multiplication. Now, we turn to the problem left unsolved in Mantero and Tonge [7]: under what circumstances is $\ell_r \tilde{\otimes} \ell_p \tilde{\otimes} \ell_q$ a Banach algebra under Schur multiplication? We give further instances when this is not a Banach algebra. To do this, we use the previous results to show that the following is true for appropriate values of p , q , and r :

For each positive B , it is possible to find integers M , N , and K and an $A = (a_{ijk}) \in \ell_r^M \tilde{\otimes} \ell_p^N \tilde{\otimes} \ell_q^K$, with each $a_{ijk} = \pm 1$ for which $\|A * A\|_{rpq} > B \|A\|_{rpq}^2$.

For this, it is important to note that, trivially, if $A = (a_{ijk}) \in \ell_r^M \tilde{\otimes} \ell_p^N \tilde{\otimes} \ell_q^K$ has each $a_{ijk} = \pm 1$, then

$$\|A * A\|_{rpq} = M^{1/r} N^{1/p} K^{1/q}. \quad (3.1)$$

PROPOSITION 3.1. *Let $1 < \min(p, q) \leq 2 \leq \max(p, q) < \infty$ and $2 \leq r < \infty$. Then $\ell_r \check{\otimes} \ell_p \check{\otimes} \ell_q$ is not a Banach algebra under Schur multiplication when*

$$\frac{1}{\min(p, q)} < \frac{2}{r} \cdot \frac{1}{\max(p, q)} + \frac{1}{2}. \quad (3.2)$$

PROOF. We consider the case where $1 < p \leq 2 \leq q < \infty$. The other case is similar.

Fix $B > 0$. By (3.1) and Corollary 2.2, it is enough to show that we can find positive integers M , N , and K with

$$M^{1/r} N^{1/p} K^{1/q} > BC \max \left(M^{2/r} N^{(2/p)-1}, N^{2/p} K^{(2/q)(1-2/r)}, N^{(2/p)-(2/r)} K^{(2/r)+(2/q)(1-2/r)} \right), \quad (3.3)$$

where C is a fixed positive number, independent of B , M , N , or K . We can achieve this with $N = K = M^t$, where $t > 0$ provided that

$$\begin{aligned} M^{t/q} &> BCM^{(1/r)+t((1/p)-1)}, \\ M^{1/r} &> BCM^{t((1/p)+(1/q)-(4/qr))}. \end{aligned} \quad (3.4)$$

Inequalities (3.4) hold simultaneously for large M if there is a $t > 0$ that satisfies

$$t \left(\frac{1}{p} + \frac{1}{q} - \frac{4}{qr} \right) < \frac{1}{r} < t \left(\frac{1}{q} - \frac{1}{p} + 1 \right). \quad (3.5)$$

Such a positive t exists if and only if

$$\frac{1}{p} < \frac{2}{r} \cdot \frac{1}{q} + \frac{1}{2}. \quad (3.6)$$

□

This result is illustrated in Figure 1. Mantero and Tonge [7] showed that, for $2 \leq r < \infty$, $\ell_r \check{\otimes} \ell_p \check{\otimes} \ell_q$ is a Banach algebra under Schur multiplication in the diagonally shaded region, but is not in the horizontally shaded region. Our results assert that $\ell_r \check{\otimes} \ell_p \check{\otimes} \ell_q$ is not a Banach algebra under Schur multiplication in the heavily shaded region. We use the same shading conventions in all subsequent figures.

If we change the role of the indices p , q , and r , we obtain the following result which is illustrated and compared to existing knowledge in Figure 2.

PROPOSITION 3.2. *Let $1 < r \leq 2 \leq p, q < \infty$. Then $\ell_r \check{\otimes} \ell_p \check{\otimes} \ell_q$ is not a Banach algebra under Schur multiplication when*

$$\frac{1}{p} \cdot \frac{1}{q} > \frac{1}{2r} - \frac{1}{4}. \quad (3.7)$$

Next, we make use of Corollary 2.4.

PROPOSITION 3.3. *Let $2 \leq p, q, r < \infty$. Then $\ell_r \check{\otimes} \ell_p \check{\otimes} \ell_q$ is not a Banach algebra under Schur multiplication when*

$$\frac{1}{p} + \frac{1}{q} > \frac{1}{2}, \quad \frac{1}{q} + \frac{1}{r} > \frac{1}{2}, \quad \text{or} \quad \frac{1}{r} + \frac{1}{p} > \frac{1}{2}. \quad (3.8)$$

Figure 1

Figure 2

Figure 3

Figure 4

Figure 5

PROOF. Fix $B > 0$. By (3.1) and Corollary 2.4, it is enough to show that we can find positive integers M , N , and K , with

$$M^{1/r} N^{1/p} K^{1/q} > BC \max \left(M^{2/r}, N^{(2/r)+(2/p)(1-2/r)} K^{(2/q)(1-2/r)}, N^{(2/p)(1-2/r)} K^{(2/r)+(2/q)(1-2/r)} \right), \quad (3.9)$$

where C is a fixed positive constant, independent of B , M , N , or K . We can achieve this with $N = K = M^t$, where $t > 0$ provided that

$$\begin{aligned} M^{t(1/p+1/q)} &> BCM^{1/r}, \\ M^{1/r} &> BCM^{(2t/r)+t(1/p+1/q)(1-4/r)}. \end{aligned} \quad (3.10)$$

Inequalities (3.10) hold simultaneously for large M if there is a $t > 0$ that satisfies

$$t \left(\frac{2}{r} + \left(\frac{1}{p} + \frac{1}{q} \right) \left(1 - \frac{4}{r} \right) \right) < \frac{1}{r} < t \left(\frac{1}{p} + \frac{1}{q} \right). \quad (3.11)$$

Such a positive t exists if and only if

$$\frac{1}{p} + \frac{1}{q} > \frac{1}{2}. \quad (3.12)$$

The other results follow in a similar manner when the roles of p , q , and r are permuted. $\square$

The results in Proposition 3.3 are illustrated and compared to previous knowledge in Figures 3 and 4. The special case when $r = 2$ is worth recording separately in Figure 5.

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