CONVEX ISOMETRIC FOLDING

E. M. ELKHOLY

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ABSTRACT. We introduce a new type of isometric folding called “convex isometric folding.” We prove that the infimum of the ratio $\text{Vol} \, N / \text{Vol} \, \varphi(N)$ over all convex isometric foldings $\varphi : N \rightarrow N$, where $N$ is a compact 2-manifold (orientable or not), is $1/4$.

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1. Introduction. A map $\varphi : M \rightarrow N$, where $M$ and $N$ are $C^\infty$ Riemannian manifolds of dimensions $m$ and $n$, respectively, is said to be an isometric folding of $M$ into $N$ if and only if for any piecewise geodesic path $y : J \rightarrow M$, the induced path $\varphi \circ y : J \rightarrow N$ is a piecewise geodesic and of the same length. The definition is given by Robertson [4].

Let $p : M \rightarrow N$ be a regular locally isometric covering and let $G$ be the group of covering transformations of $p$. An isometric folding $\varphi \in \mathcal{H}^1(M,N)$ is said to be $p$-invariant if and only if for all $g \in G$ and all $x \in X$, $p(\varphi(x)) = p(\varphi(g,x))$. See Robertson and Elkholy [5]. The set of all invariant isometric foldings is denoted by $\mathcal{H}^1_i(M,p)$.

DEFINITION 1.1. Let $\varphi \in \mathcal{H}(M,N)$, where $M$ and $N$ are $C^\infty$ Riemannian manifolds of dimensions $m$ and $n$, respectively. We say that $\varphi$ is a convex isometric folding if and only if $\varphi(M)$ can be embedded as a convex set in $\mathbb{R}^n$.

We denote the set of all convex isometric foldings of $M$ into $N$ by $C(M,N)$, and if $C(M,N) \neq \emptyset$, then it forms a subsemigroup of $\mathcal{H}(M,N)$.

DEFINITION 1.2. We say that $\varphi \in \mathcal{H}_i(M,p)$ is a $p$-invariant convex isometric folding if and only if $\varphi(M)$ can be embedded as a convex set in $\mathbb{R}^m$.

We denote the set of $p$-invariant convex isometric foldings of $M$ by $C_i(M,p)$. If $C_i(M,p) \neq \emptyset$, then for any covering map, $p : M \rightarrow N$, $C_i(M,p)$ is a subsemigroup of $C(M)$.

To solve our main problem we need the following:

(1) Robertson and Elkholy [5] proved that if $N$ is an $n$-smooth Riemannian manifold, $p : M \rightarrow N$ is its universal covering, and $G$ is the group of covering transformations of $p$, then $\mathcal{H}(N)$ is isomorphic as a semigroup to $\mathcal{H}_i(M,p)/G$.

(2) Elkholy [1] proved that if $N$ is an $n$-smooth Riemannian manifold, $p : M \rightarrow N$ is its universal covering, and $\varphi \in \mathcal{H}(N)$ such that $\varphi : \pi_1(N) \rightarrow \pi_1(N)$ is trivial, then the
corresponding folding $\psi \in \mathcal{F}_i(M, p)$ maps each fiber of $p$ to a single point.

(3) Elkholy and Al-Ahmady [3] proved that under the same conditions of (2), if $N$ is a compact 2-manifold, then

$$\frac{\text{Vol} N}{\text{Vol} \varphi(N)} = \frac{\text{Vol} F}{\text{Vol} \psi(F)},$$

(1.1)

where $F$ is a fundamental region of $G$ in $M$.

2. Convex isometric folding and covering spaces. The next theorem establishes the relation between the set of convex isometric folding of a manifold, $C(N)$, and the set of $p$-invariant convex isometric folding of its universal covering space, $C_i(M, p)$.

**Theorem 2.1.** Let $N$ be a manifold and $p : M \to N$ its universal covering. Let $G$ be the group of covering transformations of $p$. If $C(N) \neq \emptyset$, then $C(N)$ is isometric as a semigroup to $C_i(M, p)/G$.

**Proof.** Let $C(N) \neq \emptyset$. Then by using (1), there exists an isomorphism $f$ from $\mathcal{F}_i(M, p)/G$ into $\mathcal{F}(N)$. Since $C_i(M, p)$ is a subsemigroup of $\mathcal{F}_i(M, p)$, $C_i(M, p)/G$ is a subsemigroup of $\mathcal{F}_i(M, p)/G$.

Let $h = f | (C_i(M, p)/G)$. Since $C_i(M, p)/G$ is a semigroup, $h$ is a homeomorphism and also it is one-one. To show that $h$ is an onto map, we suppose that $\varphi \in C(N)$.

Hence, $\varphi \in \mathcal{F}(N)$ and, consequently, there exists $\psi \in \mathcal{F}_i(M, p)/G$. Since $\varphi \in C(N)$, $\varphi$ is trivial and hence for all $x \in M$, $\psi(Gi, x) = \psi(x)$, and therefore $\psi \in C_i(M, p)/G$.

**Theorem 2.2.** Let $N$ be a compact orientable 2-manifold and consider the universal covering space $(\mathbb{R}^2, P)$ of $N$. Let $\varphi \in C(N)$ and $\psi \in C_i(\mathbb{R}^2, p)$. Then for all $x, y \in \mathbb{R}^2$, $d(\psi(x), \psi(y)) \leq \Delta$, where $\Delta$ is the radius of a fundamental region for the covering space.

**Proof.** Elkholy [1] proved the truth of the theorem for $N = S^2$. So, we have to prove it for the connected sum of $n$-tori. First, let $N = T$ be a torus homomorphic to the quotient space obtained by identifying opposite sides of a square of length “a” as shown in Figure 1(a).

![Figure 1(a)](image)

![Figure 1(b)](image)
Suppose that \( \varphi : T \to T \) is a convex isometric folding. Then \( \varphi_\ast (\pi_1 (T)) \) is trivial. By Theorem 2.1, there exists a convex isometric folding \( \psi : \mathbb{R}^2 \to \mathbb{R}^2 \) such that for all \( x, y \in \mathbb{R}^2 \) and for all \( g \in G \), \( p (\psi(x)) = p (\psi(g \cdot x)) \). Equivalently, for all \( (P, Q) \in \mathbb{R}^2 \) and for all \( g \in \mathbb{Z} \times \mathbb{Z} \), there exists a unique \( h \in \mathbb{Z} \times \mathbb{Z} \) such that \( h \circ \psi(P, Q) = \psi(g(P, Q)) \), i.e.,

\[
\psi(P, Q) + (\sqrt{2} \Delta m, \sqrt{2} \Delta n') = \psi(P + \sqrt{2} \Delta m, Q + \sqrt{2} \Delta n), \quad \text{where } m, n, m', n' \in \mathbb{Z}.
\]

(2.1)

Consider any fundamental region \( F \) of the covering space \((\mathbb{R}^2, p)\) of \( T \), i.e., a closed square of length “a” with sides identified as shown in Figure 1(b). Since \( \varphi_\ast \) is trivial, by (2), for all \( x \in \mathbb{R}^2 \), \( \psi(G \cdot x) = \psi(x) \). Now, let \( x \) and \( y \) be distinct points of \( \mathbb{R}^2 \) such that \( x = g \cdot y \) for all \( g \in G \) and let \( d(x, y) = \alpha_1 \). Then there exists a point \( x^* = g \cdot x \) such that

\[
d(y, x^*) = \alpha_i, \quad \alpha_i = d(y, g_i \cdot x), \quad i = 1, \ldots, 4.
\]

(2.2)

Thus, there are always four equivalent points \( g_i \cdot x \), \( i = 1, \ldots, 4 \) which form the vertices of a square of length “a” and such that \( d(g_i \cdot x, y) \leq 2 \Delta \). From Figure 1(b), it is clear that \( \max d(x^*, y) \leq \Delta \) and since \( \psi \) is an isometric folding, by Robertson [4],

\[
d(\psi(x), \psi(y)) \leq d(g \cdot x, y), \quad i.e.,
\]

\[
d(\psi(x), \psi(y)) = d(\psi(g_i \cdot x), \psi(y)) \leq d(g_i \cdot x, y) = d(x, y) \leq \Delta,
\]

(2.3)

and this proves the theorem for \( N = T \).

Now, consider the connected sum of two tori, obtained as a quotient space of an octagon with sides identified as shown in Figure 2(a). The group of covering transformations \( G \) is isometric to \( \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \). Using the same previous technique, we can obtain four equivalent points as the vertices of a square of diameter \( 2 \Delta \) such that \( \max d(y, x^*) \leq \Delta \), and the result follows. This theorem, by using the above method, is true for the connected sum of \( n \)-tori.

□
Theorem 2.3. Let \( N \) be a compact nonorientable 2-manifold and consider the universal covering space \((M,p)\) of \( N \). Let \( \phi \in C(N) \) and \( \psi \in C_i(M,p) \). Then for all \( x,y \in M \), \( d(\psi(x),\psi(y)) \leq \Delta \), where \( \Delta \) is the radius of a fundamental region for the covering space.

Proof. By Elkholy [2], the theorem is true for \( N = p^2 \) and \( M = S^2 \). Now, consider the connected sum of two projective planes, the Klein bottle \( K \), homeomorphic to the quotient space obtained by identifying the opposite sides of a square as shown in Figure 3(a).

\[
\begin{align*}
\psi(P,Q) + (\sqrt{2}\Delta m', \sqrt{2}\Delta n') &= \psi(P + \sqrt{2}\Delta m, \sqrt{2}\Delta n + (-)^m Q), \quad \text{where } m, n, m', n' \in \mathbb{Z}.
\end{align*}
\]

(2.4)

Any fundamental region \( F \) of the covering space \((\mathbb{R}^2, p)\) of \( K \) is a closed square of diameter \( 2\Delta \) with the boundary identified as shown in Figure 3(b). Since \( \varphi_* \) is trivial, for all \( x \in \mathbb{R}^2 \), \( \varphi(G \cdot x) = \varphi(x) \).

Now, let \( x \) and \( y \) be distinct points of \( \mathbb{R}^2 \) such that \( y \neq g \cdot x \) for all \( g \in G \), and let \( d(x,y) = \alpha_1 \). Thus, there exists a point \( x^* = g \cdot x \) such that

\[
d(y, x^*) = \min(\alpha_i), \quad \alpha_i = d(y, g_i \cdot x), \quad i = 1,\ldots,4.
\]

(2.5)

Thus, there are always four equivalent points \( g_i \cdot x \) which form the vertices of a parallelogram such that the shortest diameter is of length less than \( 2\Delta \).

Now, the point \( y \) is either inside or on the boundary of a triangle of vertices \( g_1 \cdot x = x, g_2 \cdot x, g_3 \cdot x \). Let \( y' \) be a point equidistant from the vertices of this triangle, i.e.,

\[
d(y', x) = d(y', g_2 \cdot x) = d(y', g_3 \cdot x).
\]

(2.6)
From Figure 3(b), it is clear that \( d(y',x) < \Delta \) and, hence, \( d(x^*,y) < \Delta \). Therefore,

\[
d(\psi(x),\psi(y)) = d(\psi(g_i \cdot x),\psi(y)) \leq d(g \cdot x_i,y) = d(x^*,y) < \Delta
\]

and the result follows.

Now, let \( N \) be the connected sum of three projective planes obtained as the quotient space of a hexagon with the sides identified in pairs as indicated in Figure 4(a). In this case, \((\mathbb{R}^2,p)\) is the universal cover of \( N \) and \( G \approx \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_2 \). Using the same method as that used above, we can always have equivalent points \( g_i \cdot x, i = 1,\ldots,4 \) which form the vertices of a parallelogram whose shortest diameter is of length less than \( 2\Delta \). From Figure 4(b), we can see that \( \max d(y,x^*) < \Delta \) and the theorem is proved.

In general and by using the same technique, the theorem is also true for the connected sum of \( n \)-projective planes.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4.png}
\caption{Figure 4.}
\end{figure}

3. Volume and convex folding. The following theorem succeeds in estimating the maximum volume we may have if we convexly folded a compact 2-manifold into itself.

**Theorem 3.1.** The infimum of the ratio

\[
e_N = \frac{\text{Vol} N}{\text{Vol} \varphi(N)},
\]

where \( N \) is a compact 2-manifold over all convex isometric foldings \( \varphi \in C(N) \) of degree zero, is 4.

**Proof.** Robertson [4] has shown that if \( N \) is a compact 2-manifold, and \( \varphi : N \rightarrow N \) is a convex isometric folding, any convex isometric folding is an isometric folding, then \( \deg \varphi \) is \( \pm 1 \) or 0. We consider only the case for which \( \deg \varphi \) is zero otherwise \( \varphi(N) \) cannot be embedded as a convex subset of \( \mathbb{R}^2 \) unless \( N \) is. In this case, the set of singularities of \( \varphi \) decomposes \( N \) into an even number of strata, say \( k \), each of which is homeomorphic to \( \varphi(N) \) and, hence,

\[
\text{Vol} N = k \text{Vol} \varphi(N),
\]
that is, \( e_N \) should be an even number. To calculate the exact value of \( e_N \), consider first an orientable 2-compact manifold \( N \). By using (1.1)

\[
e_N = \frac{\text{Vol} F}{\text{Vol} \varphi(F)}
\]

(3.3)

and this means that \( e_N \) can be calculated by calculating the volume of \( F \) and of its image \( \varphi(F) \), but \( F \) is a closed square of diameter \( 2\Delta \) and \( \varphi(F) \) is a closed subset of \( F \) such that the distance \( d(x, x') \) between any two points \( x, x' \in \varphi(F) \) is at most \( \Delta \). The supremum of 2-dimensional volume of such set is \( \phi(\Delta/2)^2 \) and, hence, \( 2 < e_N \).

But \( e_N \) is an even number. Hence, \( e_N = 4 \).

Now, let \( N \) be a nonorientable 2-compact manifold, i.e., a connected sum of \( n \)-projective planes. Elkholy [2] proved the theorem for \( n = 1 \).

The fundamental region in this case is a square or a rectangle of diameter \( 2\Delta \) according to whether \( n \) is even or odd. If \( n \) is an even number, then

\[
\text{Vol} F = 2\Delta^2
\]

(3.4)

and the result follows. Now, let \( n \) be an odd number. Then \( F \) is a rectangle of lengths \( ((n+1)/2)a, ((n-1)/2)a \) and hence

\[
\text{Vol} F = 4\Delta^2 \sin \theta \cos \theta = 4\Delta^2 \frac{a(n+1)/2}{a\sqrt{(n^2+1)/2}} \frac{a(n-1)/2}{a\sqrt{(n^2+1)/2}} = \frac{n^2-1}{n^2+1} 2\Delta^2.
\]

(3.5)

Therefore, \( e_N > 2 \) for all \( n > 1 \). Since \( e_N \) is an even number, \( e_N = 4 \).

\[\square\]

REFERENCES


Elkholy: Department of Mathematics, Faculty of Science, Tanta University, Tanta, Egypt