HOW TO GENERATE ALL INTEGRAL TRIANGLES CONTAINING A GIVEN ANGLE

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ABSTRACT. We present an elementary prescription based on the *rational secant method* for generating all the integral triangles containing a given angle of rational cosine. This is a direct generalization of the ancient problem of finding all the Pythagorean triples. As an example, we discuss a specific equation studied by Diophantus of Alexandria, which turns out to be equivalent to the problem of finding all the integral triangles containing a 60° angle. The material developed here is elementary enough for inclusion in undergraduate courses and advanced high school courses.

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1. Introduction. Let θ denote the measure of a given angle such that $0^\circ < \theta < 180^\circ$. By the law of cosines, there exists a triangle containing the given angle and possessing sides of lengths r, s, t if and only if

$$r^2 + s^2 - 2rs \cos(\theta) = t^2, \quad (1.1)$$

where the side of length t is assumed to be opposite to the given angle θ .

The triangle is said to be *integral* if the side lengths r, s, t are positive integers. Since no integer solutions of (1.1) are possible unless $\cos(\theta)$ is a rational number, we may assume from here on that $\alpha = \cos(\theta)$ is rational. A solution of (1.1) in positive integers r, s, t is said to be *primitive* if the three integers r, s, t share no simultaneous common factor $g \geq 2$, then the associated integral triangle is also said to be *primitive*. Setting $u = r/t$ and $v = s/t$, our problem reduces to finding the positive rational solutions of the equation

$$u^2 + v^2 - 2\alpha uv = 1. \quad (1.2)$$

PROPOSITION 1.1. *The positive rational solutions of (1.2) are in one-to-one correspondence with the set of primitive integral solutions of (1.1).*

PROOF. On one hand, let r, s, t be a primitive integral solution of (1.1). Then the corresponding positive rational solution of (1.2) is given by $u = r/t$ and $v = s/t$, and it is easily verified that this mapping is one-to-one. On the other hand, let $u = a/b$ and $v = c/d$, in lowest terms, be a positive rational solution of (1.2), and let $k = \text{lcm}(b, d)$. It is easily verified that the inverse of the mapping $(r, s, t) \rightarrow (r/t, s/t)$ assigns to (u, v) the primitive integral solution of (1.1) given by $r = ku$, $s = kv$, and $t = k$. $\square$

2. The rational secant method. Setting $x = u - \alpha v$, $y = v$, equation (1.2) becomes

$$x^2 + (1 - \alpha^2)y^2 = 1. \quad (2.1)$$

Hence, the rational solutions of (1.2) are now in one-to-one correspondence with the rational solutions of (2.1).

In terms of x and y , the positivity constraints $u \geq 0$ and $v \geq 0$ become $x + \alpha y \geq 0$ and $y \geq 0$. Hence, the positive rational solutions of (1.2) correspond to the rational points on the part of the ellipse $\varepsilon : x^2 + (1 - \alpha^2)y^2 = 1$ which lies above the x -axis and to the right of the line $y = -(1/\alpha)x$, as depicted in Figure 2.1.

FIGURE 2.1.

Next, we proceed to derive an explicit parametric formula for all the rational points on the ellipse ε . Our derivation exploits the *rational secant method*, whose essence is captured in the following proposition.

PROPOSITION 2.1. *Let P_0 be a given rational point on the ellipse $\varepsilon : x^2 + (1 - \alpha^2)y^2 = 1$ and let P be any other point on ε . Then the slope of the secant line $\overline{P_0P}$ is a rational number if and only if P is a rational point.*

PROOF. Let μ denote the slope of $\overline{P_0P}$. If the point P is rational, then it is easy to see that μ has to be rational. Conversely, if μ is rational, let $P_0 = (x_0, y_0)$ and $P = (x, y)$, then

$$\mu x - y = \mu x_0 - y_0. \quad (2.2)$$

On the other hand, by (2.1), we have

$$(1 - \alpha^2)(y^2 - y_0^2) = x_0^2 - x^2, \quad (2.3)$$

which, when combined with the relation $y - y_0 = \mu(x - x_0)$, gives

$$-\mu(1 - \alpha^2)(y + y_0) = x_0 + x. \quad (2.4)$$

Now, equations (2.2) and (2.4) amount to a system of two linear equations in two unknowns:

$$\mu x - y = \mu x_0 - y_0, \quad x + \alpha y = -x_0 - \alpha y_0, \quad (2.5)$$

where, for simplicity, we have set $v = \mu(1 - \alpha^2)$. To extract from (2.5) the rational solutions for x and y , it remains only to show that the determinant of the system is different from zero. The determinant of the system is $\mu v + 1 = \mu^2(1 - \alpha^2) + 1$ and is different from zero because $\mu^2(1 - \alpha^2) \geq 0$. $\square$

Note that the special point $P_0 = (-1, 0)$ is a rational point on the ellipse ε . Hence, by Proposition 2.1, all of the rational points P on ε are in one-to-one correspondence with the lines $\overline{P_0P}$ of rational slope. Referring to Figure 2.1, we see that in order to conform with the positivity constraints $x + \alpha y \geq 0$ and $y \geq 0$, it is necessary and sufficient to consider all the lines $\overline{P_0P}$ with rational slopes μ in the range

$$0 < \mu < \frac{1}{1 - \alpha}. \quad (2.6)$$

Solving the system (2.5) with $(x_0, y_0) = (-1, 0)$, we obtain the following explicit formula for all rational solutions of (2.1) in terms of the parameter μ :

$$x = \frac{1 - \mu v}{1 + \mu v}, \quad y = \frac{2\mu}{1 + \mu v}, \quad (2.7)$$

where, as before, $v = \mu(1 - \alpha^2)$. Restoring u and v , we obtain

$$u = \frac{1 - \mu v + 2\alpha\mu}{1 + \mu v}, \quad v = \frac{2\mu}{1 + \mu v}. \quad (2.8)$$

If we assume that $\alpha = a/b$ and $\mu = m/n$, as fractions in lowest terms, then

$$u = \frac{b^2 n^2 - (b^2 - a^2) m^2 + 2abmn}{b^2 n^2 + (b^2 - a^2) m^2}, \quad v = \frac{2b^2 mn}{b^2 n^2 + (b^2 - a^2) m^2}, \quad (2.9)$$

from which we obtain the following (not necessarily primitive) integral solution of (1.1),

$$\begin{aligned} r' &= b^2 n^2 - (b^2 - a^2) m^2 + 2abmn, \\ s' &= 2b^2 mn, \\ t' &= b^2 n^2 + (b^2 - a^2) m^2. \end{aligned} \quad (2.10)$$

To derive from (2.10) the primitive integral solution of (1.1) corresponding to (u, v) , it remains only to reduce r' , s' , and t' by their greatest common divisor. We summarize our main result as follows.

PROPOSITION 2.2. *Let m, n be a pair of relatively prime nonnegative integers satisfying condition (2.6), with $\mu = m/n$. Let r' , s' , and t' be given as in (2.10), and let $g = \gcd(r', s', t')$. Then the primitive integral solution of (1.1), determined by (m, n) , is*

$$r = \frac{r'}{g}, \quad s = \frac{s'}{g}, \quad t = \frac{t'}{g} \quad (2.11)$$

and every primitive integral solution of (1.1) is generated in this manner.

3. An example from antiquity. In *Arithmetica*, Book IV.10, Diophantus of Alexandria seeks positive rational solutions to the equation $u^3 + v^3 = u + v$. Dividing both sides by $u + v$, the equation reduces to $u^2 + v^2 - uv = 1$, which is a special case of (1.2)

with $\alpha = 1/2$. Thus, the problem is equivalent to finding primitive integral triangles containing a 60° angle. Applying (2.10) with $a = 1$, $b = 2$, and $0 < m < 2n$, we get

$$r' = 4n^2 - 3m^2 + 4mn, \quad s' = 8mn, \quad t' = 4n^2 + 3m^2. \quad (3.1)$$

So, by Proposition 2.2, all the primitive integral triangles containing a 60° angle are given by $r = r'/g$, $s = s'/g$, $t = t'/g$, where $g = \gcd(r', s', t')$. Diophantus offers the particular solution $u = 5/7$ and $v = 8/7$, which corresponds to taking $m = 1$ and $n = 1$ in (3.1).

4. Exercises. (1) In the 60° case (above example), suppose that $m = 2^p m_1$ and $n = 3^q n_1$, where m_1 is odd and n_1 is not a multiple of 3. Show that

$$\gcd(r', s', t') = \begin{cases} 1, & \text{if } p = 0 \text{ and } q = 0, \\ 16, & \text{if } p = 1 \text{ and } q = 0, \\ 4, & \text{if } p \geq 2 \text{ and } q = 0, \\ 3, & \text{if } p = 0 \text{ and } q \geq 1, \\ 48, & \text{if } p = 1 \text{ and } q \geq 1, \\ 12, & \text{if } p \geq 2 \text{ and } q \geq 1. \end{cases} \quad (4.1)$$

(2) In *Arithmetica*, Book IV.11, Diophantus seeks positive rational solutions to $u^3 - v^3 = u - v$. What is the corresponding problem stated in terms of primitive integral triangles? Determine all the solutions and derive an explicit formula for $\gcd(r', s', t')$ that is analogous to (4.1).

(3) The simplest primitive integral triangle containing a 60° angle is the equilateral triangle with sides $r = s = t = 1$. Find the values of m and n in (3.1) corresponding to this triangle.

(4) Determine all the curves $F(x, y) = c$ in the plane for which the rational secant method holds. Can you formulate an analogous proposition for the curves in 3-space? Can you formulate a generalization to varieties of higher dimension, say surfaces in 3-space?

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