A GENERALIZATION OF A THEOREM OF FAITH AND MENAL AND APPLICATIONS

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ABSTRACT. In 1995, Faith and Menal have established the V -ring theorem which gives a characterization of a V -ring. In this paper, we generalize this theorem to V -modules and consider some applications for Noetherian self-cogenerators.

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1. Preliminaries. Throughout this paper, R denotes an associative ring with identity and all modules considered are unitary right R -modules. Homomorphisms are written on the side opposite to that of scalars. For any module M , the sum of all simple submodules of M is called a *socle* of M and is denoted by $\text{Soc}(M)$. Dually, the intersection of all maximal submodules of M is called a *radical* of M and is denoted by $\text{Rad}(M)$. $(R)_n$ denotes the $n \times n$ matrix ring over R . Let M be a module. An *M -generated module* is a module which is isomorphic to a factor module of $M^{(I)}$ for some index set I . We denote by $\sigma[M]$ the full subcategory of $\text{Mod-}R$ whose objects are all submodules of M -generated modules, and by $E_M(N)$ the *M -injective hull* of a module N in $\sigma[M]$ which is the trace of M in $E(M)$, where $E(M)$ indicates the injective hull of M , that is $E_M(N) = \sum \{f(M) : f \in \text{Hom}_R(M, E(N))\}$ in $\sigma[M]$ (see Wisbauer [9, 17.9, (2)]). A module M is called a *V -module* if every proper submodule of M is an intersection of maximal submodules of M or, equivalently, if every simple module (in $\sigma[M]$ or $\text{Mod-}R$) is M -injective (see, e.g., Wisbauer [9, 23.1]). A ring R is called a *right V -ring* if R is a V -module when considered as a right module over itself, i.e., every simple module is injective. For notation, definitions and, familiar results concerning the ring theory we mainly follow Anderson and Fuller [2] and Wisbauer [9].

2. A generalization of a theorem of Faith and Menal. Let M, E be modules, and $M_E^* = \text{Hom}_R(M, E)$. For each subset Z of M_E^* and each subset X of M , the right annihilator in M is denoted by $r_M(Z)$, and the left annihilator in M_E^* is denoted by $\ell_{M_E^*}(X)$, that is,

$$r_M(Z) = \{m \in M : Zm = 0\}, \quad \ell_{M_E^*}(X) = \{f \in M_E^* : fX = 0\}. \quad (2.1)$$

In [6], Faith and Menal showed that a ring R is a right V -ring if and only if there exists a semisimple module W such that $I = r_R \ell_W(I)$ for every right ideal I of R . In this case, we say that W satisfies the double annihilator condition (d.a.c.) with respect to right ideals. This characterization of a V -ring by the existence of a duality between the right

ideals via annihilation and submodules of a semisimple module is called the *V-ring theorem*. A ring R is called a *right Johns ring* if R is right Noetherian and satisfies that any right ideal is a right annihilator ideal. It is known that a right Johns ring is not right Artinian (see Faith and Menal [4]). If a ring R is right Johns, then $\tilde{I} = r_{R/J(R)} \ell_{\text{Soc}(R)}(\tilde{I})$ for any right ideal $\tilde{I}$ of $R/J(R)$, that is $R/J(R)$ is a right *V-ring* by the *V-ring theorem*, where $J(R)$ denotes the Jacobson radical of R (see Faith and Menal [6]). We begin with the following theorem.

THEOREM 2.1. *Let M be a module. Then the following are equivalent:*

- (1) M is a *V-module*;
- (2) *there exists a semisimple module W satisfying $N = r_L \ell_{L_W^*}(N)$ for any module L in $\text{Mod-}R$ and any submodule N of L such that L/N is in $\sigma[M]$;*
- (3) *there exists a semisimple module W' in $\sigma[M]$ satisfying $N = r_L \ell_{L_{W'}^*}(N)$ for any module L in $\sigma[M]$ and any submodule N of L .*

PROOF. (1) $\Rightarrow$ (2), (1) $\Rightarrow$ (3). Let $\{S_i\}_{i \in \Omega}$ be an irredundant set of representatives of the simple modules in $\sigma[M]$. Then, $\bigoplus_{i \in \Omega} E_M(S_i)$ is the minimal M -injective cogenerator of $\sigma[M]$ (see Wisbauer [9, p. 143]).

Since M is a *V-module*, $E_M(S_i) = S_i$ for each $i \in \Omega$ and, hence, $\bigoplus_{i \in \Omega} S_i$ is a semisimple cogenerator of $\sigma[M]$. Hence, $\bigoplus_{i \in \Omega} S_i$ cogenerates L/N for any module L and any submodule $N \subseteq L$ such that L/N is in $\sigma[M]$. By Albu and Năstăsescu [1, Prop. 3.5], $\bigoplus_{i \in \Omega} S_i$ cogenerates the factor module L/N if and only if $N = r_L \ell_{\bigoplus_{i \in \Omega} S_i^*}(N)$. Now, the proof of (1) $\Rightarrow$ (2) is clear.

Since $\bigoplus_{i \in \Omega} S_i$ is in $\sigma[M]$ and, for any module L in $\sigma[M]$, each factor module of L belongs to $\sigma[M]$, the implication (1) $\Rightarrow$ (3) also follows from the proof above.

(2) $\Rightarrow$ (1), (3) $\Rightarrow$ (1). For a semisimple module W satisfying condition (2), since each factor module of M belongs to $\sigma[M]$, we see that $N = r_M \ell_{M_W^*}(N)$ holds for any submodule N of M . Hence, $M/N \rightarrow W^{\ell_{M_W^*}(N)}, m + N \mapsto (f(m))_{f \in \ell_{M_W^*}(N)}$ is an R -monomorphism. This readily implies that $\text{Rad}(M/N) = 0$. Hence, N is an intersection of maximal submodules of M . Thus, M is a *V-module*. For a semisimple module W' satisfying condition (3), it also follows from the same argument above that M is a *V-module*. $\square$

COROLLARY 2.2. *Let M be a module. Then the following statements are equivalent:*

- (1) M is a *V-module*;
- (2) *there exists a semisimple module W satisfying $I = r_R \ell_W(I)$ for any right ideal I of R such that R/I is in $\sigma[M]$;*
- (3) *there exists a semisimple module W' in $\sigma[M]$ satisfying $N = r_M \ell_{M_{W'}^*}(N)$ for any submodule N of M .*

In this case, W and W' cogenerate any module in $\sigma[M]$.

PROOF. (1) $\Rightarrow$ (2), (1) $\Rightarrow$ (3). These are obvious by Theorem 2.1.

(3) $\Rightarrow$ (1). Follows immediately from the same argument of (3) $\Rightarrow$ (1) in the proof of Theorem 2.1.

(2) $\Rightarrow$ (1). Let S be any simple module in $\sigma[M]$. To show that S is M -injective, we need to show that S is N -injective for every cyclic submodule N of M by Wisbauer [9, 16.3, (b)]. So, let N be a cyclic submodule of M and let f be a nonzero R -homomorphism

from a submodule N' of N to S . Since N is cyclic, $N \cong R/I$ for some right ideal I of R and, hence, $N' \cong L/I$ for some right ideal L of R . Therefore, $\text{Ker}(f) \cong L'/I$ for some right ideal $L' \subset L$ of R . Since $N, \text{Ker}(f)$ are in $\sigma[M]$ and since $\sigma[M]$ is closed under cokernels, R/L' is in $\sigma[M]$. The hypothesis implies that $L' = r_R \ell_W(L')$. By [1, Prop. 3.5], there is an exact sequence $0 \rightarrow R/L' \rightarrow W^Y$ for some set Y . This readily implies that $\text{Rad}(R/L') = 0$. Then since L' is an intersection of maximal right ideals, there is a maximal right ideal K of R such that $K \supseteq L'$ but $K \not\supseteq L$. Since $N'/\text{Ker}(f) \cong L/L'$ is simple, it follows that $L \cap K = L'$. Then $R/I/K/I \cong R/K = (L+K)/K \cong (L/L \cap K) = L/L' \cong N'/\text{Ker}(f) \cong S$ and, therefore, f can be extended to an $\tilde{f}$ in $\text{Hom}_R(N, S)$. Hence, S is N -injective and M is a V -module.

Finally, we show that a semisimple module W satisfying condition (2) and a semisimple module W' satisfying condition (3) cogenerate any module in $\sigma[M]$. For any maximal right ideal I with R/I in $\sigma[M]$, we observe that $I = r_R \ell_W(I)$ holds. Thus, it follows, by almost same argument in the proof of the corollary in Faith and Menal [6], that W satisfying condition (2) cogenerates any module in $\sigma[M]$. Next, since $E_M(S) = S$ for any simple module S in $\sigma[M]$, $f(M) = S$ for some $f \in \text{Hom}_R(M, S)$ and, hence, $M/\text{Ker}(f) \cong S$. Then since W' satisfies the d.a.c. with respect to the submodules of M , $\text{Ker}(f) = r_M \ell_{M_{W'}^*}(\text{Ker}(f))$. Since $\text{Ker}(f)$ is maximal, $\text{Ker}(f) = r_M(g) = \text{Ker}(g)$ for some $g \in M_{W'}^*$. Therefore, W' contains a copy of S . This implies that W' satisfying condition (3) cogenerates any module in $\sigma[M]$. $\square$

REMARK 2.3. Let M be a module. If there exists a semisimple module W , which need not be in $\sigma[M]$, such that W satisfies the d.a.c. with respect to any submodule of M , then it is easy to deduce from the argument of the proof of (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1) in Theorem 2.1 that M is a V -module.

PROPOSITION 2.4. Let M be a module. If M contains a copy of each simple factor module of M , then the following statements are equivalent:

- (1) $M/\text{Rad}(M)$ is a V -module;
- (2) $\text{Soc}(M)$ cogenerates any module in $\sigma[M/\text{Rad}(M)]$;
- (3) $\tilde{I} = r_{R/J(R)} \ell_{\text{Soc}(M)}(\tilde{I})$ for any right ideal $\tilde{I}$ of $R/J(R)$ such that $(R/J(R))/\tilde{I}$ is in $\tilde{\sigma}[M/\text{Rad}(M)] = \sigma[M/\text{Rad}(M)] \cap (\text{Mod-}R/J(R))$.

PROOF. (1) $\Rightarrow$ (2). Let $\{S_i\}_{i \in \Omega}$ be an irredundant set of representatives of the simple R -modules in $\sigma[M/\text{Rad}(M)]$. Since $M/\text{Rad}(M)$ is a V -module, by Wisbauer [9, p. 143], we know that $\bigoplus_{i \in \Omega} S_i$ cogenerates any module in $\sigma[M/\text{Rad}(M)]$. So, it suffices to show that $\text{Soc}(M)$ contains a copy of S_i for each $i \in \Omega$. Since $E_{M/\text{Rad}(M)}(S_i) = S_i$, $f(M/\text{Rad}(M)) = S_i$ for some $f \in \text{Hom}_R(M/\text{Rad}(M), S_i)$. Clearly, S_i is a simple homomorphic image of M . Thus, by hypothesis, there exists an exact sequence $0 \rightarrow S_i \rightarrow \text{Soc}(M)$. Obviously, it follows that $\text{Soc}(M)$ cogenerates any module in $\sigma[M/\text{Rad}(M)]$.

(2) $\Rightarrow$ (3). We note that any module in $\tilde{\sigma}[M/\text{Rad}(M)]$ belongs to $\sigma[M/\text{Rad}(M)]$. Since $\text{Soc}(M)$ cogenerates any module in $\tilde{\sigma}[M/\text{Rad}(M)]$, again by virtue of [1, Prop. 3.5], we have $\tilde{I} = r_{R/J(R)} \ell_{\text{Soc}(M)}(\tilde{I})$ for every right ideal $\tilde{I}$ of $R/J(R)$ such that $R/J(R)/\tilde{I}$ in $\tilde{\sigma}[M/\text{Rad}(M)]$.

(3) $\Rightarrow$ (1). Note that $M/\text{Rad}(M)$ is a V -module as a right $R/J(R)$ -module if and only if $M/\text{Rad}(M)$ is a V -module as a right R -module. Since $M/\text{Rad}(M)_{R/J(R)}$ is a V -module by Corollary 2.2, $M/\text{Rad}(M)_R$ is a V -module. $\square$

Recall that a ring R is a *right Kasch ring* if any simple right R -module is isomorphic to a minimal right ideal of R . Since a ring R is right Kasch if and only if every maximal right ideal of R is a right annihilator ideal (see, e.g., Faith [3, p. 37]), we observe that a right Johns ring is right Kasch.

COROLLARY 2.5. *If a ring R is right Kasch, then the following statements are equivalent:*

- (1) $R/J(R)$ is a right V -ring;
- (2) $\text{Soc}(R)$ cogenerates any module in $\text{Mod-}R/J(R)$;
- (3) $\tilde{I} = r_{R/J(R)} \ell_{\text{Soc}(R)}(\tilde{I})$ for every right ideal $\tilde{I}$ of $R/J(R)$.

3. Applications. A module M is called a *self-generator* if M generates every submodule of M . Dually, a module M is called a *self-cogenerator* if M cogenerates every factor module of M . By Albu and Năstăseacu [1, Prop. 3.5], M is a self-cogenerator if and only if $N = r_M \ell_\Lambda(N)$ for any submodule N of M , where $\Lambda = \text{End}(M_R)$. In particular, R_R is a self-cogenerator if and only if $I = r_R \ell_R(I)$ for any right ideal I of R .

THEOREM 3.1. *Let M be a self-cogenerator and let $\Lambda = \text{End}(M_R)$. If there exists a (Λ, R) -bimodule $W \subseteq \text{Soc}(M_R)$ such that $M_W^* = \ell_\Lambda(X)$ for some subset X of M , then $\tilde{M} = M/r_M(M_W^*)$ is a V -module.*

PROOF. By virtue of Remark 2.3, we need to prove that $N/r_M(M_W^*) = r_{\tilde{M}} \ell_{\tilde{M}_W^*}(N/r_M(M_W^*))$ for every submodule $N \supseteq r_M(M_W^*)$ of M . Applying the W -dual functor $\text{Hom}_R(-, W)$ to the natural exact sequence $M \rightarrow \tilde{M} \rightarrow 0$, we get that the dual sequence $0 \rightarrow \tilde{M}_W^* \rightarrow M_W^*$ is exact. Since $\ell_\Lambda r_M(M_W^*) = \ell_\Lambda r_M \ell_\Lambda(X) = \ell_\Lambda(X) = M_W^*$ by hypothesis, we have $M_W^* \cong \tilde{M}_W^*$ as an abelian group. Since M is a self-cogenerator, there exists a subset $\{g_i\}_{i \in I} \subseteq \Lambda$ such that $N = r_M(\{g_i\}_{i \in I})$. If we take the left annihilator in Λ for $r_M(M_W^*) \subseteq N$, we have $\{g_i\}_{i \in I} \subseteq \ell_\Lambda(N) \subseteq \ell_\Lambda r_M(M_W^*) = M_W^*$. Since $M_W^* \cong \tilde{M}_W^*$ by the natural way, so that $\{\tilde{g}_i\}_{i \in I} \subseteq \tilde{M}_W^*$ follows, where $\tilde{g}_i: \tilde{M} \rightarrow W$ denotes the R -homomorphism induced by g_i for each $i \in I$. Thus, we obtain that $\{\tilde{g}_i\}_{i \in I} \subseteq \ell_{\tilde{M}_W^*}(N/r_M(M_W^*))$. So, if we note that $r_{\tilde{M}}(\{\tilde{g}_i\}_{i \in I}) = r_M(\{g_i\}_{i \in I})/r_M(M_W^*)$, then we have

$$r_{\tilde{M}} \ell_{\tilde{M}_W^*} \left(\frac{N}{r_M(M_W^*)} \right) \subseteq r_{\tilde{M}}(\{\tilde{g}_i\}_{i \in I}) = \frac{r_M(\{g_i\}_{i \in I})}{r_M(M_W^*)} = \frac{N}{r_M(M_W^*)}. \quad (3.1)$$

Since the reverse inclusion is easily verified, this completes the proof. $\square$

Observe that a right Johns ring is a trivial Noetherian self-cogenerator. Next, we consider a nontrivial module which is a Noetherian self-cogenerator. It is known that the class of right Johns rings is not Morita stable (see Faith and Menal [5, Rem. 3.7]). A ring R is called a *strongly right Johns ring* if $(R)_n$ is right Johns for all positive integers n . However, it is not known if a strongly right Johns ring must be quasi-Frobenius, equivalently, right Artinian (cf. Faith and Menal [5]). Using a right Johns ring and a strongly right Johns ring, we construct Noetherian self-cogenerators. Let $n > 0$, $S = (R)_n$ and $P = R^{(n)}$. Consider the functor $H = \text{Hom}_R(P, -) : \text{Mod-}R \rightarrow \text{Mod-}S$. We note that the functor $H = \text{Hom}_R(P, -) : \text{Mod-}R \rightarrow \text{Mod-}S$ is an equivalence.

EXAMPLE 1. Suppose that R is a strongly right Johns ring and consider $P = R^{(n)}$. Since $H(R)^n \cong H(P) \cong S$, every factor module of P_R is cogenerated by R if and only if

every factor module of S_S is cogenerated by $H(R)$ if and only if every factor module of S_S is cogenerated by S . Thus, P_R gives an example of a Noetherian self-cogenerator.

EXAMPLE 2. Suppose that R is a right Johns ring and consider $P = R^{(n)}$ as a right S -module by the usual way. By Anderson and Fuller [2, Prop. 21.7], each submodule of $H(R)_S$ is of the form $\text{Im}H(g)$ for some submodule I of R_R and the inclusion map $g : I \rightarrow R$. Since R_R is a self-cogenerator, $I = r_R \ell_R(I)$ holds for any right ideal I of R . By Kurata and Hashimoto [8, Lem. 1.19], we have $\text{Im}H(g) = r_{H(R)} \ell_R(\text{Im}H(g))$. Then, $H(R)/\text{Im}H(g) \rightarrow H(R)^{\ell_R(\text{Im}H(g))}, m + \text{Im}H(g) \mapsto (rm)_{r \in \ell_R(\text{Im}H(g))}$ is an S -monomorphism. Thus, $H(R)_S$ is a self-cogenerator. Since $P_S \cong H(R)_S$ is a natural isomorphism, $H(R)_S$ is a self-cogenerator if and only if P_S is a self-cogenerator. Thus P_S is a self-cogenerator. Since S is right Noetherian, the finitely generated module P_S is right Noetherian. Therefore, P_S gives an example of a Noetherian self-cogenerator.

PROPOSITION 3.2. *If M is a Noetherian projective self-cogenerator, then $\Lambda = \text{End}(M_R)$ is a right Johns ring and $\text{End}(M/\text{Rad}(M)_R)$ is a right V-ring.*

PROOF. Suppose that I is any finitely generated right ideal of Λ . Since M is projective, $I = \text{Hom}_R(M, IM)$ by Wisbauer [9, 18.4]. Since M is a self-cogenerator, there is some set Y of Λ such that $IM = r_M(Y)$. Now, it is straightforward to verify that

$$\text{Hom}_R(M, r_M(Y)) = r_\Lambda(Y). \quad (3.2)$$

This implies that I is a right annihilator ideal. Since M is Noetherian and projective, it follows from Albu and Năstăsescu [1, Prop. 4.12] that Λ is right Noetherian. Hence, Λ is a right Johns ring. Now, by Anderson and Fuller [2, Cor. 17.12], $\text{End}(M/\text{Rad}(M)_R) \cong \Lambda/J(\Lambda)$. Since $\Lambda/J(\Lambda)$ is a right V-ring, $\text{End}(M/\text{Rad}(M)_R)$ is a right V-ring. $\square$

COROLLARY 3.3. *Let M be a Noetherian projective self-generator and a self-cogenerator, then $M/\text{Rad}(M)$ is a V-module.*

PROOF. We note that $M/\text{Rad}(M)$ is projective and a self-generator in $\text{Mod-}R/J(R)$. By Proposition 3.2, $\text{End}(M/\text{Rad}(M)_R)$ is a right V-ring and, hence, $\text{End}(M/\text{Rad}(M)_{R/J(R)})$ is a right V-ring. Thus, by Hirano [7, Thm. 3.11], $M/\text{Rad}(M)_{R/J(R)}$ is a V-module and so $M/\text{Rad}(M)_R$ is a V-module. $\square$

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