APPROXIMATING FIXED POINTS OF NONEXPANSIVE MAPPINGS

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ABSTRACT. We consider a mapping $S$ of the form

$$S = \alpha_0 I + \alpha_1 T_1 + \alpha_2 T_2 + \cdots + \alpha_k T_k,$$

where $\alpha_i \geq 0$, $\alpha_0 > 0$, $\alpha_1 > 0$ and $\sum_{i=0}^k \alpha_i = 1$. We show that the Picard iterates of $S$ converge to a common fixed point of $T_i$ ($i = 1, 2, \ldots, k$) in a Banach space when $T_i$ ($i = 1, 2, \ldots, k$) are nonexpansive.

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1. Introduction. Let $X$ be a Banach space and $C$ a convex subset of $X$. A mapping $T : C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y$ in $C$.

Specifically, the iterative process studied by Kirk is given by

$$x_{n+1} = \alpha_0 x_n + \alpha_1 Tx_n + \alpha_2 T^2 x_n + \cdots + \alpha_k T^k x_n, \quad (1.1)$$

where $\alpha_i \geq 0$, $\alpha_1 > 0$ and $\sum_{i=0}^k \alpha_i = 1$.

Kirk [1] has investigated an iterative process for approximating fixed points of nonexpansive mapping on convex subset of a uniformly convex Banach space. Recently, Maiti and Saha [2] improved the result of Kirk.

Let $T_i : C \to C$ ($i = 1, 2, \ldots, k$) be nonexpansive mappings, and let

$$S = \alpha_0 I + \alpha_1 T_1 + \alpha_2 T_2 + \cdots + \alpha_k T_k, \quad (1.2)$$

where $\alpha_i \geq 0$, $\alpha_0 > 0$, $\alpha_0 > 0$ and $\sum_{i=0}^k \alpha_i = 1$.

In this paper, we show that the Picard iterates (1.2) of $S$ converge to a common fixed point of $T_i$ ($i = 1, 2, \ldots, k$) in a Banach space, without any assumption on convexity of Banach space. This result generalizes the corresponding result of Kirk [1], Maiti and Saha [2], Senter and Dotson [4].

2. Main results

**Lemma 2.1.** Let $X$ be a normed space and $\{a_n\}$ and $\{b_n\}$ be two sequences in $X$ satisfying

(i) $\lim_{n \to \infty} \|a_n\| = d$,

(ii) $\limsup_{n \to \infty} \|b_n\| \leq d$ and $\{\sum_{i=1}^n b_i\}$ is bounded,

(iii) there is a constant $t$ with $0 < t < 1$ such that $a_{n+1} = (1 - t)a_n + tb_n$ for all $n \geq 1$. Then $d = 0$. 
**Proof.** Suppose that \( d > 0 \) and it follows from (ii) that \( \sum_{i=1}^{n+m-1} b_i \) is bounded for all \( n \) and \( m \). Let

\[
M = \sup \left\{ \left\| \sum_{i=n}^{n+m-1} b_i \right\| : n, m = 1, 2, 3, \ldots \right\}.
\]  

(2.1)

Choose a number \( N \) such that

\[
N > \max \left( \frac{2tM}{d}, 1 \right).
\]  

(2.2)

We can choose a positive \( \varepsilon \) such that

\[
1 - 2\varepsilon \exp \left( \frac{N+1}{1-t} \right) > \frac{1}{2}.
\]  

(2.3)

By \( 0 < t < 1 \), there exists a natural \( k \) such that

\[
N < kt \leq N + 1.
\]  

(2.4)

Since \( \lim_{n \to \infty} \|a_n\| = d \), \( \limsup_{n \to \infty} \|b_n\| \leq d \) and \( \varepsilon \) independent of \( n \), without loss of generality we may assume that, for all \( n \geq 1 \),

\[
d(1 - \varepsilon) < \|a_n\| < d(1 + \varepsilon) \quad \text{and} \quad \|b_n\| < d(1 + \varepsilon).
\]  

(2.5)

Setting \( s = 1 - t \) from (iii), we obtain by induction

\[
a_{k+1} = s^k a_1 + ts^{k-1} b_1 + \cdots + ts b_{k-1} + tb_k, \quad a_{k+1} \in B := \text{co} \{a_1, b_1, b_2, \ldots, b_k\}.
\]  

(2.6)

Let \( x = (1/k) \sum_{i=1}^{k} b_i \) and

\[
y = \frac{s^k}{1-s^k} \{ a_1 + t[s^{-1} - (kt)^{-1}] b_1 + t[s^{-2} - (kt)^{-1}] b_2 + \cdots + t[s^{-k} - (kt)^{-1}] b_k \}.
\]  

(2.7)

Then it is clear that \( x, y \in B \) and \( a_{k+1} = s^k x + (1-s^k)y \). Therefore,

\[
d(1 - \varepsilon) < \|a_{k+1}\| \leq s^k \|x\| + (1-s^k)\|y\| \leq s^k \|x\| + (1-s^k)d(1+\varepsilon).
\]  

(2.8)

Hence, we have

\[
\|x\| > d(1-s^{-k}(2-s^k)\varepsilon) > d(1-2\varepsilon s^{-k})
\]

\[
= d \left\{ 1 - 2\varepsilon \exp \left[ \sum_{i=1}^{k} \log \left( 1 + \frac{t}{1-t} \right) \right] \right\} \geq d \left[ 1 - 2\varepsilon \exp \left( \sum_{i=1}^{k} \frac{t}{1-t} \right) \right]
\]

\[
= d \left[ 1 - 2\varepsilon \exp \left( \frac{kt}{1-t} \right) \right] \geq d \left[ 1 - 2\varepsilon \exp \left( \frac{N+1}{1-t} \right) \right] > d/2,
\]  

(2.9)

since \( \log(1+u) \leq u \) for \( -1 < u < \infty \).

On the other hand, we have

\[
\|x\| = \frac{1}{k} \left\| \sum_{i=1}^{k} b_i \right\| \leq \frac{M}{k} \leq \frac{d}{2M} M = \frac{d}{2},
\]  

(2.10)

arriving at a contradiction. This completes the proof. \( \square \)
**Lemma 2.2.** Let $C$ be a subset of a normed space $X$ and $T_n : C \to C$ be a nonexpansive mapping for all $n = 1, 2, \ldots, k$. If for an arbitrary $x_0 \in C$ and $\{x_n\}$ is defined by (1.2), then
\[
\|x_{n+1} - p\| \leq \|x_n - p\| \tag{2.11}
\]
for all $n \geq 1$ and $p \in F(T)$, where $F(T)$ denotes the common fixed point set of $T_i$ $(i = 1, 2, \ldots, k)$.

**Proof.** Since $p = S p$ for all $p \in F(T)$ and $T_i$ $(i = 1, 2, \ldots, k)$ is nonexpansive, we have
\[
\|x_{n+1} - p\| = \|S x_n - S p\| \leq \sum_{i=1}^{k} \alpha_i \|T_i x_n - T_i p\| = \|x_n - p\| \tag{2.12}
\]
for all $n \geq 1$ and all $p \in F(T)$. This completes the proof. \qed

**Theorem 2.3.** Let $C$ be a nonempty closed convex and bounded subset of a Banach space $X$ and $T_i : C \to C$ $(i = 1, 2, \ldots, k)$ be nonexpansive mappings. If for an arbitrary $x_0 \in C$ and $\{x_n\}$ is defined by (1.2), then $\|x_n - S x_n\| \to 0$ as $n \to \infty$.

**Proof.** By (1.2) and $T_i$ is nonexpansive mapping, we have
\[
\|x_{n+1} - S x_{n+1}\| \leq \|S x_n - S x_{n+1}\|
\leq \alpha_0 \|x_n - x_{n+1}\| + \sum_{i=1}^{k} \alpha_i \|T_i x_n - T_i x_{n+1}\| \leq \|x_n - S x_n\|. \tag{2.13}
\]
Hence $\|x_n - S x_n\| \to d$ as $n \to \infty$.

Set $a_n = x_n - S x_n$, $b_n = 1/(1 - \alpha_0) \sum_{i=1}^{k} \alpha_i (T_i x_n - T_i x_{n+1})$, we have $a_{n+1} = \alpha_0 a_n + (1 - \alpha_0) b_n$ and
\[
\|b_n\| \leq \frac{1}{1 - \alpha_0} \sum_{i=1}^{k} \alpha_i \|T_i x_n - T_i x_{n+1}\| \leq \frac{1}{1 - \alpha_0} \sum_{i=1}^{k} \alpha_i \|x_n - x_{n+1}\| = \|a_n\|. \tag{2.14}
\]
Since $\lim_{n \to \infty} \|a_n\| = \lim_{n \to \infty} \|x_n - S x_n\| = d$,
\[
\limsup_{n \to \infty} \|b_n\| \leq d. \tag{2.15}
\]
Finally, we have
\[
\left\| \sum_{j=1}^{n} b_j \right\| = \left\| \sum_{j=1}^{n} \left[ \frac{1}{1 - \alpha_0} \sum_{i=1}^{k} \alpha_i (T_i x_j - T_i x_{j+1}) \right] \right\|
= \frac{1}{1 - \alpha_0} \left\| \sum_{i=1}^{k} \alpha_i \left[ \sum_{j=1}^{n} (T_i x_j - T_i x_{j+1}) \right] \right\|
= \frac{1}{1 - \alpha_0} \left\| \sum_{i=1}^{k} \alpha_i (T_i x_1 - T_i x_{n+1}) \right\|
\leq \frac{1}{1 - \alpha_0} \sum_{i=1}^{k} \alpha_i \|T_i x_1 - T_i x_{n+1}\| \leq \|x_1 - x_{n+1}\|. \tag{2.16}
\]
Then $\|\sum_{j=1}^{n} b_j\|$ is bounded. Setting $t = 1 - \alpha_0$, then $a_{n+1} = (1-t)a_n + tb_n$ and $0 < t < 1$. It follows from Lemma 2.1 that $d = 0$, this completes the proof.

Recall that a Banach space $X$ is said to satisfy Opial’s condition [3] if the condition $x_n \to x_0$ weakly implies

$$\limsup_{n \to \infty} \|x_n - x_0\| < \limsup_{n \to \infty} \|x_n - y\|$$

(2.17)

for all $y \neq x_0$.

**Theorem 2.4.** Let $X$ be a Banach space which satisfies Opial’s condition, $C$ be weakly compact and convex, and let $T_i$ ($i = 1, 2, \ldots, k$) and $\{x_n\}$ be as in Theorem 2.3. Then $\{x_n\}$ converges weakly to a fixed point of $S$.

**Proof.** Due to weak compactness of $C$, there exists $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to a $p \in C$. With standard proof we show that $p = Sp$. We suppose that $\{x_n\}$ does not converge weakly to $p$; then there are $\{x_{n_l}\}$ and $q \neq p$ such that $x_{n_l} \to q$ weakly and $q = Sq$. By Theorem 2.3 and Opial’s condition of $X$, we have

$$\lim_{n \to \infty} \|x_n - p\| = \lim_{j \to \infty} \|x_{n_j} - p\| < \lim_{j \to \infty} \|x_{n_j} - q\|$$

$$= \lim_{l \to \infty} \|x_{n_l} - q\| < \lim_{l \to \infty} \|x_{n_l} - p\| = \lim_{n \to \infty} \|x_n - p\|,$$

(2.18)

a contradiction. This completes the proof.

Let $D$ be a subset of a Banach space $X$. Mappings $T_i : D \to X$ ($i = 1, 2, \ldots, k$) with a nonempty common fixed points set $F(T)$ in $D$ will be said to satisfy condition $A$ [2, 4] if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for $r \in (0, \infty)$, such that $\|x - Sx\| \geq f(d(x, F(T)))$ for all $x \in D$, where $S$ is defined by (1.2), $d(x, F(T)) = \inf \{\|x - z\| : z \in F(T)\}$.

**Theorem 2.5.** Let $X$, $C$, and $\{x_n\}$ be as in Theorem 2.3. Let $T_i : C \to X$ ($i = 1, 2, \ldots, k$) be nonexpansive mappings with a nonempty common fixed points set $F(T)$ in $C$. If $T_i$ satisfies condition $A$, then $\{x_n\}$ converges to a member of $F(T)$.

**Proof.** By condition $A$, we have

$$\|x_n - Sx_n\| \geq f\{d[x_n, F(T)]\}$$

(2.19)

for all $n \geq 0$. Since $\{d[x_n, F(T)]\}$ is decreasing by Lemma 2.2, it follows from Theorem 2.3 that

$$\lim_{n \to \infty} \{d[x_n, F(T)]\} = 0.$$  

(2.20)

We can thus choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\|x_{n_j} - p_j\| < 2^{-j}$$

(2.21)

for all integers $j \geq 1$ and some sequence $\{p_j\}$ in $F(T)$. Again by Lemma 2.2, we have $\|x_{n_j+1} - p_j\| \leq \|x_{n_j} - p_j\| < 2^{-j}$, and hence

$$\|p_{j+1} - p_j\| \leq \|p_{j+1} - x_{n_{j+1}}\| + \|x_{n_{j+1}} - p_j\| \leq 2^{-j+1} + 2^{-j} < 2^{-j+1},$$

(2.22)
which show that \( \{p_j\} \) is Cauchy and therefore converges strongly to a point \( p \) in \( F(T) \) since \( F(T) \) is closed. Now it is readily seen that \( \{x_{n_j}\} \) and hence \( \{x_n\} \) itself, by Lemma 2.2, converges strongly to \( p \).

**Remark 2.6.** Theorem 2.5 generalizes [2, 4, Theorem 2.3] to a Banach space.

**Theorem 2.7.** Let \( C \) be a closed convex subset of a Banach space \( X \), and \( T_i \) (\( i = 1, 2, \ldots, k \)) be nonexpansive mappings from \( C \) into a compact subset of \( X \). If \( \{x_n\} \) is as in Theorem 2.3, then \( \{x_n\} \) converges to a fixed point of \( S \).

**Proof.** By Theorem 2.3 and the precompactness of \( S(C) \), we see that \( \{x_n\} \) admits a strongly convergent subsequence \( \{x_{n_j}\} \) whose limit we denote by \( z \). Then, again by Theorem 2.3, we have \( z = Sz \); namely, \( z \) is a fixed point of \( S \). Since \( \|x_n - z\| \) is decreasing by Lemma 2.1, \( z \) is actually the strong limit of the sequence \( \{x_n\} \) itself.

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