Internat. J. Math. & Math. Sci.
Vol. 23, No. 3 (2000) 223–224
S0161171200003446
© Hindawi Publishing Corp.

A NOTE ON COMMUTATIVITY OF NONASSOCIATIVE RINGS

M. S. S. KHAN

(Received 31 December 1998 and in revised form 16 May 1999)

ABSTRACT. A theorem on commutativity of nonassociate ring is given.

Keywords and phrases. Rings with unity, commutativity of rings, nonassociative rings.

2000 Mathematics Subject Classification. Primary 17A30; Secondary 16Y30.

In 1968, Johnsen, Outcalt, and Yaqub [3] have established that a nonassociative ring R with identity 1 satisfying the relation $(xy)^2 = x^2y^2$ for every x and y in R, is commutative. Gupta [2] has shown that if R is a nonassociative 2-torsion free ring with unity 1 satisfying $(xy)^2 = (yx)^2$ for all x, y in R, then R is commutative. Later, Yuanchun [4] proved that a Baer-semisimple ring R is commutative if and only if $(xy)^2 - xy^2x$ is central. The existence of noncommutative ring R with $R^2 \subseteq Z(R)$, center of R, rules out the possibility that $(xy)^2 - xy^2x \epsilon Z(R)$ might yield commutativity even in associative rings. As an example, consider $A_3 = \{(a_{ij})/a_{ij}$ are integers with $a_{ij} = 0,\ i \geq j\}$. Then A_3 is a noncommutative nilpotent ring of index 3 in which $(xy)^2 - xy^2x$ is central for all x, y in A_3.

This naturally gives rise to the following question: what additional conditions are needed to insure the commutativity of R when R is an arbitrary ring? With this motivation, Ashraf, Quadri, and Zelinsky [1] established the following result.

THEOREM 1. *Let R be an associative ring with unity 1 satisfying $(xy)^2 = yx^2y$ for all x, y in R, then R is commutative.*

They used very complicated combinatorial arguments. In this connection we prove the following results.

THEOREM 2. *Let R be a nonassociative ring with unity 1 satisfying $(xy)^2 = (xy^2)x$ for all x, y in R. Then R is commutative.*

PROOF. Replacing $y+1$ for y in $(xy)^2 = (xy^2)x$, we obtain

$$\big(x(y+1)\big)^2 = \big(x(y+1)^2\big)x, \quad \text{which yields } x(xy) = (xy)x. \tag{1}$$

Repeating this argument for $x+1$ in place of x, equation (1) gives

$$x(xy) + xy = (xy)x + yx. \tag{2}$$

Thus equation (2) together with equation (1), shows that R is commutative.

Similarly, we can prove the following theorem. □

Theorem 3. *Let R be a nonassociative ring with unity 1 satisfying $(xy)^2 = (yx^2)y$ for all x, y in R. Then R is commutative.*

If we drop the restriction of unity 1 in the hypothesis, R may be badly noncommutative.

Example. Let

$$R = \left\{ \alpha I + B \;\middle|\; I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ B = \begin{pmatrix} 0 & \beta & \gamma \\ 0 & 0 & \delta \\ 0 & 0 & 0 \end{pmatrix},\ \alpha, \beta, \gamma, \delta \in Z_p \right\}, \tag{3}$$

p is a prime such that p/n if n odd or $2p/n$ if n even, and Z_p is the ring of integers modulo p. Then $B^3 = 0$, for $n \geq 3$ and

$$(\alpha I + B)^n = \alpha^n I + n\alpha^{n-1} B + \frac{n(n-1)}{2!} \alpha^{n-2} B^2 + \cdots = \alpha^n I, \tag{4}$$

because $n = 0$ and $n(n-1)/2! = 0$ in Z_p, where p/n and $2p/n(n-1)$.

However, R need not be commutative.

Acknowledgement. The author is thankful to the learned referee for the useful comments.

References

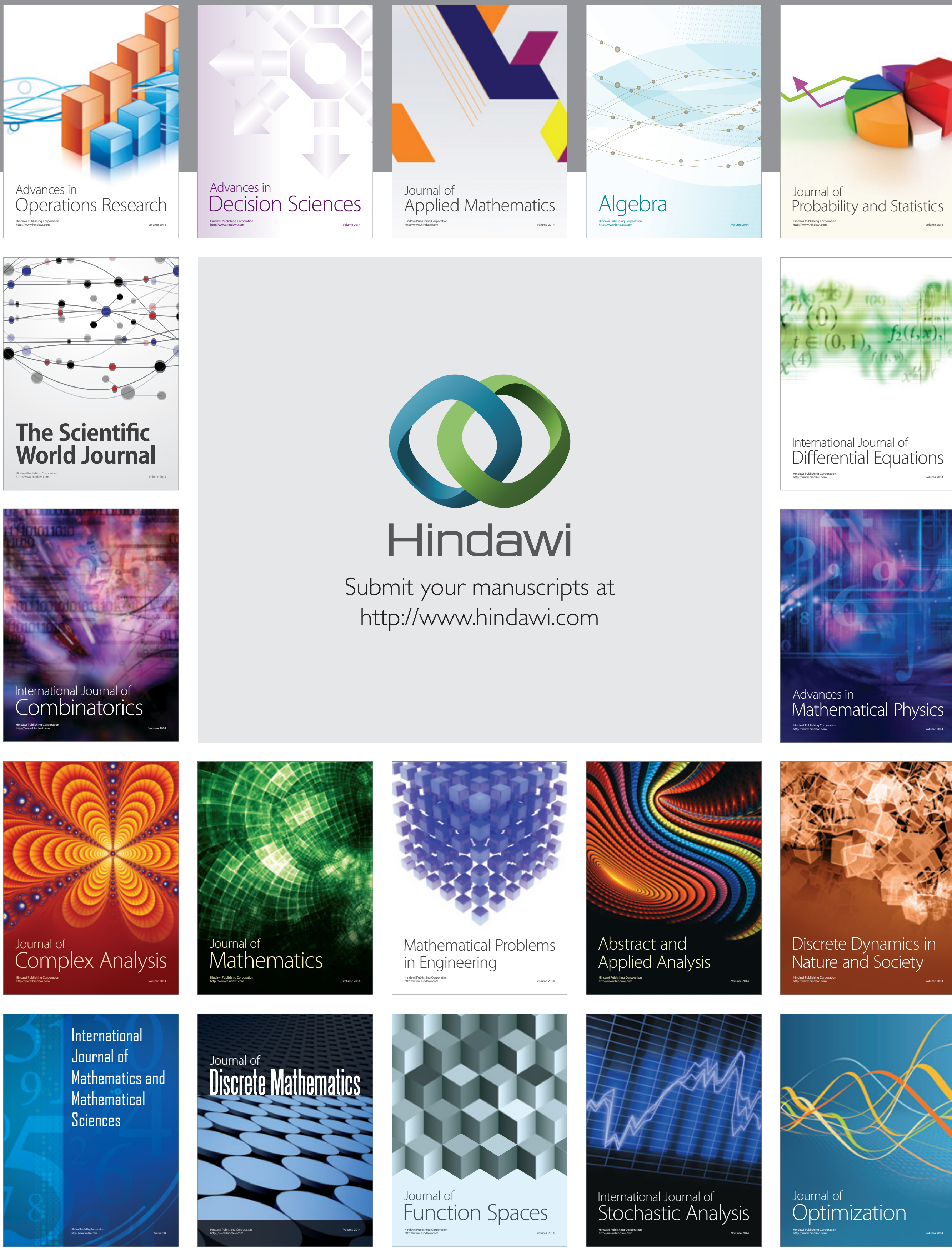
[1] M. Ashraf, M. A. Quadri, and D. Zelinsky, *Some polynomial identities that imply commutativity for rings*, Amer. Math. Monthly **95** (1988), no. 4, 336–339. MR 89c:16045. Zbl 643.16021.

[2] R. N. Gupta, *A note on commutativity of rings*, Math. Student **39** (1971), 184–186. MR 48 6192. Zbl 271.17003.

[3] E. C. Johnsen, D. L. Outcalt, and A. Yaqub, *An elementary commutativity theorem for rings*, Amer. Math. Monthly **75** (1968), 288–289. MR 37#1417. Zbl 162.33602.

[4] G. Yuanchun, *Some commutativity theorems of rings*, Acta Sci. Natur. Univ. Jilin **3** (1983), 11–18.

Khan: Department of Mathematics and Computer Science University of Leicester, Leicester, LE1 7RH, England, UK



Advances in Operations Research
Advances in Decision Sciences
Journal of Applied Mathematics
Algebra
Journal of Probability and Statistics
The Scientific World Journal
International Journal of Differential Equations
Hindawi
Submit your manuscripts at http://www.hindawi.com
International Journal of Combinatorics
Advances in Mathematical Physics
Journal of Complex Analysis
Journal of Mathematics
Mathematical Problems in Engineering
Abstract and Applied Analysis
Discrete Dynamics in Nature and Society
International Journal of Mathematics and Mathematical Sciences
Journal of Discrete Mathematics
Journal of Function Spaces
International Journal of Stochastic Analysis
Journal of Optimization