STABILITY OF SECOND-ORDER RECURRENCES MODULO p^r

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ABSTRACT. The concept of sequence *stability* generalizes the idea of uniform distribution. A sequence is *p-stable* if the set of residue frequencies of the sequence reduced modulo p^r is eventually constant as a function of r . The authors identify and characterize the stability of second-order recurrences modulo odd primes.

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1. Introduction. Let $w(a, b) = (w)$ be a second-order linear recurrence satisfying the relation

$$w_{n+2} = aw_{n+1} - bw_n, \quad (1.1)$$

where the parameters a and b and the initial terms w_0 and w_1 are all rational integers. If m is a positive integer, then the sequence $w(a, b)$ is eventually periodic when reduced modulo m . For any residue d , we let $v_w(d, m)$ denote the number of times that the residue d appears in one shortest period (cycle) of the recurrence $w(a, b)$ modulo m . The function $v_w(d, m)$ is the *frequency distribution* function of the sequence $w(a, b)$ modulo m . Let $\Omega_w(m)$ be the image of the frequency distribution function, i.e.,

$$\Omega_w(m) = \{v_w(d, m) \mid d \in \mathbb{Z}\}. \quad (1.2)$$

We are concerned here with the possible values taken on by the frequency distribution function $v_w(d, m)$ when $m = p^r$ is a power of an odd prime.

In 1992, while investigating the Fibonacci sequence $u(1, -1)$ modulo powers of two, Eliot Jacobson [12] discovered that the frequency sets $\Omega_{u(1, -1)}(2^r)$ are eventually constant as a function of r . This observation led to the definition of sequence stability.

DEFINITION 1.1. A sequence (w) is *stable modulo p*, or simply *p-stable*, if there is a positive integer N such that $\Omega_w(p^r) = \Omega_w(p^N)$ for all $r \geq N$.

Our interest in sequence stability developed naturally from earlier study of frequency distributions of second-order recurrence sequences. In the 1970s, an extensive investigation of second-order recurrence sequences led to the complete characterization, by Bumby [1] and Webb and Long [22], of second-order recurrence sequences for which $|\Omega(m)| = 1$. The frequency distribution function of these sequences is constant and they are called *uniformly distributed*. Investigation of distributions for which $|\Omega(m)|$ is small soon followed.

In 1988 and 1989, Jacobson [10, 11] recognized that, although the Fibonacci sequence $u(1, -1)$ is not always uniformly distributed modulo p , the set $\Omega_{u(1, -1)}(p)$ is often small. He studied moduli m for which $u(1, -1)$ modulo m is *almost uniform*, i.e., $|\Omega(m)| = 2$. Conjectures proposed at the First Meeting of the Canadian Number Theory Association in Banff (1988) spurred Andrzej Schinzel [14] to classify the sets $\Omega_w(p)$ for a large class of second-order recurrences (w) and odd primes p for which $|\Omega(m)| \leq 4$.

With the introduction of the concept of stability, the study of the frequency distributions of second-order recurrence sequences modulo prime powers has become much more tractable. Once a sequence is identified as p -stable, the set of allowable frequencies can, in theory, be computed with a finite computation; the frequency distributions modulo arbitrary powers of p can then be determined. In practice, as Carlip and Jacobson observed in [4], these computations may be arbitrarily long; the sets $\Omega(p^r)$ may be arbitrarily large and the constant N (the *index of stability*) required in the definition of stability also arbitrarily large.

Stability of second-order recurrences modulo two has been extensively studied by Carlip and Jacobson in [2, 3, 4, 5], while stability modulo odd primes has been examined by Carlip, Jacobson, and Somer in [6] and Carroll, Jacobson, and Somer in [9]. In recent work Carlip and Somer [7, 21] have studied the frequency distributions of second-order recurrences modulo powers of odd primes. The primary purpose of this paper is to show how the results in [7] and [21] can be applied to characterize the stability of sequences. In particular, we exhibit several classes of second-order recurrences that fail to be p -stable and provide explicit new criteria for other second-order recurrence sequences to be p -stable. In the process we extend earlier results and provide a catalogue of what is currently known about the p -stability of second-order recurrences for odd p .

2. Preliminaries and notation. We make free use of the terminology and notation of [7] and [21]. For the convenience of the reader, we provide some of the basic definitions and specialized results here.

2.1. The family $\mathcal{F}(a, b)$. Throughout this paper, we fix a prime p , usually odd, and study the p -stability of second-order recurrences from a family $\mathcal{F}(a, b)$ of second-order recurrences $w(a, b) = (w)$ that satisfy the recurrence relation

$$w_{n+2} = aw_{n+1} - bw_n, \quad (2.1)$$

for various initial terms w_0 and w_1 .

If $p^m \parallel (w_0, w_1)$ for some $m \geq 1$, then $p^m \parallel (w_n, w_{n+1})$ for all $n \geq 0$. If (w'_n) is the recurrence defined by $w'_n = w_n/p^m$, then $p \nmid (w'_0, w'_1)$ and $v_{w'}(d, p^r) = v_w(p^m d, p^{r+m})$ for all $r \geq 1$. Thus, we can determine the frequency distribution function of (w) from that of (w') , and accordingly we restrict our attention to those recurrences for which $p \nmid (w_0, w_1)$.

DEFINITION 2.1. The family $\mathcal{F}(a, b)$ consists of all second-order recurrence sequences (w) that satisfy (2.1) and $p \nmid (w_0, w_1)$.

In general, elements w_n for which $p \mid w_n$ behave quite differently from elements

for which $p \nmid w_n$. We refer to elements w_n for which $p \mid w_n$ as p -singular elements of (w) and elements for which $p \nmid w_n$ as p -regular elements of (w) . Analogously, we call an integer d p -singular if $p \mid d$ and p -regular if $p \nmid d$.

In addition to the constants a and b , there are other parameters associated with the family $\mathcal{F}(a, b)$ and referred to as *global parameters* of the family. These include constants associated with the *characteristic polynomial*

$$f(x) = x^2 - ax + b, \quad (2.2)$$

such as the roots α and β and the discriminant $D = D(a, b) = a^2 - 4b$. A number of our results require constraints on D , e.g., requiring that D be p -regular or a quadratic residue modulo p .

2.2. Stability and the stability index. As mentioned in the introduction, a sequence (w) is p -stable if there is a positive integer N such that $\Omega_w(p^r) = \Omega_w(p^N)$ for all $r \geq N$. In [4], Carlip and Jacobson observed that when $p = 2$, the integer N , the *generation* at which stability *begins*, may be arbitrarily large. We formalize the study of the parameter N with the following definition.

DEFINITION 2.2. Suppose that (w) is p -stable. The smallest positive integer N such that $\Omega_w(p^r) = \Omega_w(p^N)$ for all $r \geq N$ is called the *index of p -stability*, or simply the *index of stability* when p is understood. The stability index of (w) is denoted by $\iota_w(p)$, or simply $\iota(p)$ when (w) is understood.

2.3. Blocks of sequences. The family $\mathcal{F}(a, b)$ is endowed with a natural equivalence relation that preserves many important properties.

DEFINITION 2.3. The recurrence $w'(a, b)$ is a *multiple of a translation (mot)* of $w(a, b)$ modulo p^r if there exist integers m and c such that $p \nmid c$ and for all n

$$w'_n \equiv cw_{n+m} \pmod{p^r}. \quad (2.3)$$

The equivalence classes of the relation **mot** are called the p^r -blocks. If d is any integer, then $\nu_w(d, p^r) = \nu_{w'}(cd, p^r)$, and therefore for every n

$$\nu_w(w_{n+m}, p^r) = \nu_{w'}(w'_n, p^r). \quad (2.4)$$

Thus, two sequences in the same block have the same *pattern* of frequencies of residues in corresponding cycles.

2.4. Periods, restricted periods, and multipliers. If the defining parameter b is p -regular, then each sequence $w(a, b)$ is purely periodic when reduced modulo p^r . We let $\lambda_w(p^r)$ denote the *period* of $w(a, b)$ modulo p^r , i.e., the least positive integer λ such that for all n

$$w_{n+\lambda} \equiv w_n \pmod{p^r}. \quad (2.5)$$

Similarly, $h_w(p^r)$ denotes the *restricted period* of $w(a, b)$ modulo p^r , i.e., the least positive integer h such that for some integer M and for all n

$$w_{n+h} \equiv Mw_n \pmod{p^r}. \quad (2.6)$$

The integer $M = M_w(p^r)$, defined up to congruence modulo p^r , is called the *multiplier*

of $w(a, b)$ modulo p^r . It is well known that $h_w(p^r) \mid \lambda_w(p^r)$ and that $E_w(p^r) = \lambda_w(p^r)/h_w(p^r)$ is the multiplicative order in $(\mathbf{Z}/p^r\mathbf{Z})^*$ of the multiplier $M_w(p^r)$.

2.5. Regular recurrences. In this paper, we are primarily concerned with p -regular sequences. A recurrence sequence $w(a, b)$ is *regular* modulo p , or simply p -regular, if

$$\begin{vmatrix} w_0 & w_1 \\ w_1 & w_2 \end{vmatrix} = w_0w_2 - w_1^2 \not\equiv 0 \pmod{p}. \quad (2.7)$$

It is evident that p -regularity is preserved by the equivalence relation **mot**. Thus, if a block contains a regular recurrence, then every recurrence in that block is regular and we refer to that block as a *regular block*.

If $p \mid (w_0, w_1)$, then certainly (w) is not p -regular. The second-order recurrence sequences that fail to be p -regular may be characterized as those sequences that, modulo p , satisfy a recurrence relation of order one.

A straightforward argument shows that all p -regular recurrences in $\mathcal{F}(a, b)$ have the same period, restricted period, and multiplier modulo p^r . Consequently, these may be considered to be global parameters of the family $\mathcal{F}(a, b)$, and we use the notation $\lambda(p^r)$, $h(p^r)$, and $M(p^r)$ to represent the period, restricted period, and multiplier modulo p^r of all p -regular recurrences in $\mathcal{F}(a, b)$. We make frequent use of the quotient $\lambda(p)/h(p)$, a global parameter that we now recognize as the multiplicative order of the multiplier $M(p)$ corresponding to any p -regular sequence in $\mathcal{F}(a, b)$. For notational convenience we define $s = E(p) = \lambda(p)/h(p)$.

We require Lemma 2.4, which characterizes the restricted period in terms of the characteristic roots.

LEMMA 2.4. *Suppose that $p \nmid D(a, b)$ and that α and β are the roots of the characteristic polynomial $f(x) = x^2 - ax + b$. Let P be a prime ideal lying over p in $\mathbf{Q}(\alpha)$. Then $h(p^r)$ is the least integer n such that $\alpha^n \equiv \beta^n \pmod{P^r}$.*

PROOF. This follows from the standard Binet formula for the regular sequence $u(a, b)$ (defined in Definition 2.5). See, e.g., [6, Lem. 2.1]. $\square$

2.6. Some special recurrences. Three special sequences in the family $\mathcal{F}(a, b)$ play a prominent role in our study. These sequences, (u) , (v) , and (t) , are characterized by their initial terms.

DEFINITION 2.5. (a) The Lucas sequence of the first kind (LSFK), $u(a, b)$, is the sequence in $\mathcal{F}(a, b)$ with initial terms $u_0 = 0$ and $u_1 = 1$.

(b) The Lucas sequence of the second kind (LSSK), $v(a, b)$, is the sequence in $\mathcal{F}(a, b)$ with initial terms $v_0 = 2$ and $v_1 = a$.

(c) The recurrence $t(a, b)$, defined when p is odd, $(\frac{b}{p}) = 1$, and $u(a, b)$ has even restricted period modulo p , is the recurrence in $\mathcal{F}(a, b)$ with initial terms $t_0 = 1$ and $t_1 = \theta$, where $\theta^2 \equiv b \pmod{p}$ and $0 \leq \theta \leq (p-1)/2$.

If in place of θ , in the definition of $t(a, b)$, we use the square root θ' of b modulo p satisfying $(p-1)/2 \leq \theta' \leq p-1$, then, by [20, pp. 534–535], the resulting sequence is a **mot** of $t(a, b)$ modulo p . Moreover, the same paper shows that when $t(a, b)$ is defined, it is never a **mot** of $u(a, b)$ or of $v(a, b)$ modulo p .

We make frequent use of the fact that the recurrence $u(a, b)$ is always p -regular. It follows that $\lambda(p^r) = \lambda_u(p^r)$, $h(p^r) = h_u(p^r)$, and $M(p^r) \equiv M_u(p^r) \pmod{p^r}$. Moreover, $M(p^r) \equiv u_{h+1} \pmod{p^r}$, and $h(p^r)$ is the smallest index h such that $u_h \equiv 0 \pmod{p^r}$. Further, we note that the recurrence $v(a, b)$ is p -regular if and only if $p \nmid D(a, b)$ and that $t(a, b)$ is p -regular whenever $t(a, b)$ is defined.

We require Lemma 2.6, which relates the p -blocks containing the sequences $u(a, b)$ and $v(a, b)$.

LEMMA 2.6. *The sequences $u(a, b)$ and $v(a, b)$ lie in the same p -block if and only if $h(p)$ is even.*

PROOF. Clearly, $v(a, b)$ is a **mot** of $u(a, b)$ modulo p if and only if $p \mid v_m$ for some positive integer m . The lemma now follows from [8, pp. 42, 47]. $\square$

2.7. Nondegenerate recurrences. Given a prime p , we define the global parameter e to be the largest integer, if it exists, such that $h(p^e) = h(p)$. Since $u(a, b)$ is p -regular, it follows that e is uniquely determined by $p^e \parallel u_{h(p)}$. If e does not exist, then $u_{h(p)} = 0$, and the p -regular sequences in $\mathcal{F}$ are called *degenerate*.

Similarly, f is the largest integer such that $\lambda(p^f) = \lambda(p)$. It is easy to see that if e exists, then f also exists and $f \leq e$.

The parameters e and f play a critical role in the structure theory of second-order recurrence sequences. One of the outstanding open questions in the theory is whether for the family $\mathcal{F}(1, -1)$, the family that contains the Fibonacci sequence $u(1, -1)$, there exists a prime p for which $e > 1$.

In this paper, the relationship between e and f determines the subsequent analysis. If $p \nmid D$ and $\text{ord}_{p^{2e}}(b) \mid p - 1$, then Theorems 2.13 and 2.10 imply that $e = f$. In particular, this is true when $b = \pm 1$. On the other hand, if $e \geq 2$, then it may occur that $f < e$ or $f = e$.

2.8. Distribution theorems. Our discussion of sequence stability makes use of specialized results and notation concerning the frequency distributions of residues of second-order recurrences that appear in [7] and [21]. We list several of these key theorems here.

The principle methodology of [7] and [21] requires a subtle analysis of the ratios of certain terms of a recurrence $w(a, b)$ modulo p^r . Such ratios are well defined when the denominator is p -regular and may be viewed as embedded in the localization $\mathbf{Z}_p$ of the integers at the ideal (p) . To facilitate analysis of these ratios, we make the following definition.

DEFINITION 2.7. If (w) is a recurrence and m and n are nonnegative integers such that $p \nmid w_n$, then we define $\rho_w(n, m)$, or simply $\rho(n, m)$, to be the element $w_{n+m}/w_n \in \mathbf{Z}_p$.

We also require several special constants. We define $r^* = \max(\lceil r/2 \rceil, e)$ for use in Theorem 2.12, and, in order to handle small values of r , we define $e^* = \min(r, e)$ and $f^* = \min(r, f)$. Also, we recall that $s = E_w(p) = \lambda_w(p)/h_w(p)$ is the multiplicative order in $\mathbf{Z}/(p)$ of the multiplier $M_w(p)$.

THEOREM 2.8 [7, Thm. 6.2]. *Suppose that $w(a, b) \in \mathcal{F}(a, b)$ is p -regular, $f < e$, and $p \nmid d$. Then, for all $r > f$,*

$$v(d, p^r) = v(d, p^f) \leq v(d, p). \quad (2.8)$$

HYPOTHESIS 2.9 [7, Hypothesis 6.3]. *There exist a p -regular recurrence $w(a, b) \in \mathcal{F}(a, b)$ and an integer n such that $\text{ord}_{p^{2e}}(\rho_w(n, h(p^e))) \mid p - 1$.*

THEOREM 2.10 [7, Thm. 6.4]. *If Hypothesis 2.9 holds, then $e = f$ and*

$$\text{ord}_{p^{2e}}(\rho_w(n, h(p^e))) = s. \quad (2.9)$$

Conversely, if $e = f$ and $(\frac{D}{p}) = -1$, then Hypothesis 2.9 holds.

THEOREM 2.11 [7, Thm. 6.5]. *Let $w'(a, b) \in \mathcal{F}(a, b)$ be a p -regular recurrence satisfying the conditions of Hypothesis 2.9 and assume that $r > f$. Let $w(a, b) \in \mathcal{F}(a, b)$ and assume that $w(a, b)$ is not a **mot** of $w'(a, b)$ modulo p . Then, for all p -regular residues d modulo p^r ,*

$$v(d, p^r) = v(d, p^f) \leq v(d, p). \quad (2.10)$$

THEOREM 2.12 [7, Thm. 6.7]. *Let $w'(a, b) \in \mathcal{F}(a, b)$ be a p -regular recurrence satisfying the conditions of Hypothesis 2.9 and assume that $r > f$. Let $w(a, b) \in \mathcal{F}(a, b)$ and assume that $w(a, b)$ is a **mot** of $w'(a, b)$ modulo p . Choose m maximal such that $1 \leq m \leq e$ and $w(a, b)$ is a **mot** of $w'(a, b)$ modulo p^m .*

(a) *If $r \leq e + m$ or if $e = m$, then there exist at least s distinct p -regular residues d modulo p^r for which*

$$v_w(d, p^r) \geq p^{r-r^*}. \quad (2.11)$$

(b) *If $1 \leq m < e$ and $r > e + m$, then there exist at least $p^{r-r^*-m}s$ distinct p -regular residues d modulo p^r for which*

$$v_w(d, p^r) \geq p^m. \quad (2.12)$$

THEOREM 2.13 [7, Thm. 6.8]. *Suppose that $p \nmid D(a, b)$ and $\text{ord}_{p^{2e}}(b) \mid p - 1$. Then $v(a, b)$ satisfies the conditions of Hypothesis 2.9 for $n = 0$. In particular, Hypothesis 2.9 is true when $n = 0$ and $b = \pm 1$.*

THEOREM 2.14 [7, Thm. 6.9]. *Suppose that $w(a, b) \in \mathcal{F}(a, b)$ is a **mot** of $u(a, b)$ modulo p^{e^*} . Suppose that $p \mid d$. Then*

$$v(d, p^r) = \begin{cases} 0 & \text{if } p^{e^*} \nmid d, \\ p^{e^*-f^*}s & \text{if } p^{e^*} \mid d. \end{cases} \quad (2.13)$$

The statement and proof of Theorem 3.3 use an integer y whose definition first appeared in [7]. The parameter y plays a prominent role in the statement and proof of Theorem 2.15.

THEOREM 2.15 [21, Thm. 6.1]. *Suppose that $e > 1$ and that $w(a, b) \in \mathcal{F}(a, b)$ is a **mot** of $u(a, b)$ modulo p , but not a **mot** of $u(a, b)$ modulo p^{e^*} . Choose m maximal*

such that $w(a, b)$ is a **mot** of $u(a, b)$ modulo p^m and n minimal such that $p \mid w_n$.

If $p \mid d$ and $v(d, p^r) > 0$, then $p^m \parallel d$. Furthermore,

$$v(d, p^r) = \begin{cases} p^{r-f^*} & \text{if } m < r \leq \min(m+f, e), \\ p^m & \text{if } e-m > f \text{ and } \min(m+f, e) < r, \\ p^{e-f} & \text{if } e-m < f \text{ and } \min(m+f, e) < r. \end{cases} \quad (2.14)$$

If $e-m = f$, then

$$\text{ord}_{p^{2e-2m}} \left(\frac{w_{n+h(p^e)}/p^m}{w_n/p^m} \right) = p^\gamma s \quad (2.15)$$

for some integer γ satisfying $0 \leq \gamma \leq f$, and all possibilities for γ occur. If $\gamma \geq 1$ and $r > e$, then

$$v(d, p^r) = p^{\min(r-f, e-\gamma)}, \quad (2.16)$$

and, if $\gamma = 0$ and $r > 2e-m$, then there exists a residue d such that $p^m \parallel d$ and

$$v(d, p^r) \geq p^{r-f-\lceil (r-2e+m)/2 \rceil} = p^{r-f-\lceil (r-e-f)/2 \rceil}. \quad (2.17)$$

3. Principal results. Throughout this section, we assume that $w(a, b) \in \mathcal{F}(a, b)$ is a nondegenerate, regular second-order recurrence. We fix a prime p , assumed to be odd unless otherwise noted.

3.1. Uniform distribution. We begin with the classical result on uniform distribution of second-order recurrences of Bumby [1] and Webb and Long [22]. The sequences described in this theorem are *uniformly distributed* modulo all powers of the prime p . Since the frequency s is independent of the power of p , these sequences are p -stable.

THEOREM 3.1 (Bumby [1], Webb and Long [22]). *Let $w(a, b)$ be a second-order recurrence and p a prime, not necessarily odd. Assume that the following conditions hold:*

- (a) $p \mid D$;
- (b) $p \nmid ab$ if $p \geq 3$;
- (c) if $p = 2$, then $a \equiv 0 \pmod{2}$, $b \equiv 1 \pmod{2}$, and $w_0 + w_1 \equiv 1 \pmod{2}$;
- (d) if $p \geq 3$, then $p \nmid 2w_1 - aw_0$;
- (e) if $p = 2$ and $r \geq 2$, then $a \equiv 2 \pmod{4}$, $b \equiv 1 \pmod{4}$, and $w_0 + w_1 \equiv 1 \pmod{2}$;
- (f) if $p = 3$ and $r \geq 2$, then $a^2 \not\equiv b \pmod{9}$.

Then $w(a, b)$ is p -stable, $\iota(p) = 1$, and $\Omega(p^r) = \{s\}$ for all $r \geq 1$.

PROOF. All parts of this theorem are proved in [1] and [22]. □

3.2. The condition $e > f$. To a great degree, the p -stability of regular sequences in the family $\mathcal{F}(a, b)$ can be characterized by the relationship between the global parameters e and f . We recall that, in any case, $e \geq f$. In this section, we consider those two-term recurrence sequences for which $e > f$. We characterize the p -stability

of most of the sequences satisfying this condition: The only sequences omitted lie in the same p^e -block as $u(a, b)$.

In the first theorem, we show that such recurrences are p -stable when they contain no p -singular terms.

THEOREM 3.2. *Suppose that $e > f$. If $w(a, b)$ is not a **mot** of $u(a, b)$ modulo p , then $w(a, b)$ has no p -singular terms and is p -stable with $1 \leq \iota(p) \leq f$.*

PROOF. Since $\mathbf{Z}/(p)$ is a field, it is clear that only one p -block contains sequences with p -singular terms. Since $u(a, b)$ certainly has p -singular terms, it follows that $w(a, b)$ has no p -singular terms.

On the other hand, by Theorem 2.8, if d is p -regular and $r \geq f$, then

$$v(d, p^r) = v(d, p^f) \leq v(d, p). \quad (3.1)$$

Consequently, if $r \geq f$, then $\Omega_w(p^r) = \Omega_w(p^f)$, and hence $w(a, b)$ is p -stable with $\iota(p) \leq f$. $\square$

Next, we turn to recurrences that contain p -singular terms. As observed in the previous proof, these sequences lie in the same p -block as $u(a, b)$. If $w(a, b)$ is in the same p -block as $u(a, b)$, but not the same p^e -block, then there is a maximal positive integer m such that $1 \leq m < e$, and $w(a, b)$ lies in the same block as $u(a, b)$ modulo p^m . In Theorem 3.3, we characterize the stability of these sequences in terms of the relation of m to $e - f$. Note, in particular, that in (d) we exhibit a class of sequences that fail to be p -stable.

THEOREM 3.3. *Suppose that $e > f$. Assume that $w(a, b)$ is a **mot** of $u(a, b)$ modulo p but not modulo p^e , and choose m maximal such that $w(a, b)$ is a **mot** of $u(a, b)$ modulo p^m . If $m = e - f$, then define γ as in Theorem 2.15. Then we have the following stability criteria for $w(a, b) \in \mathcal{F}(a, b)$.*

- (a) *If $m < e - f$, then $w(a, b)$ is p -stable and $\iota(p) \leq m + f$.*
- (b) *If $m > e - f$, then $w(a, b)$ is p -stable and $\iota(p) \leq e$.*
- (c) *If $m = e - f$ and $\gamma \geq 1$, then $w(a, b)$ is p -stable and $\iota(p) \leq e + f - \gamma$.*
- (d) *If $m = e - f$ and $\gamma = 0$, then $w(a, b)$ is not p -stable.*

NOTE. The definition and existence of the parameter γ that appears in (c) and (d) is a consequence of Theorem 2.15. The reader may consult [7] and [21] for additional details.

PROOF. First, suppose that $p \nmid d$. Then, by Theorem 2.8,

$$v(d, p^r) = v(d, p^f) \leq v(d, p) \quad (3.2)$$

when $r \geq f$. In particular, (3.2) holds when $r \geq m + f$, when $r \geq e$, and, since $\gamma \leq f$, also when $r \geq e + f - \gamma$.

Next, suppose that $p \mid d$ and $v(d, p^r) > 0$. Since $e > f \geq 1$, we can easily apply Theorem 2.15 to prove (a), (b), and (c).

(a) If $m < e - f$, then Theorem 2.15 implies that

$$v(d, p^r) = p^m \quad (3.3)$$

when $r \geq m + f$. Clearly, (3.2) and (3.3) yield (a).

(b) If $m > e - f$, then Theorem 2.15 implies that

$$\nu(d, p^r) = p^{e-f} \quad (3.4)$$

when $r \geq e$. Now, (3.2) and (3.4) yield (b).

(c) If $m = e - f$ and $\gamma \geq 1$, then Theorem 2.15 implies that

$$\nu(d, p^r) = p^{e-\gamma} \quad (3.5)$$

when $r \geq e + f - \gamma$. In this case, (3.2) and (3.5) yield (c).

(d) Finally, assume that $m = e - f$ and $\gamma = 0$. By Theorem 2.15, if $r > 2e - m$, then there exists a residue d such that $p^m \mid d$ for which

$$\nu(d, p^r) \geq p^{r-f-\lceil (r-2e+m)/2 \rceil}. \quad (3.6)$$

Clearly (3.6) implies that $\max(\Omega_w(p^r))$ is unbounded as a function of r , and hence $w(a, b)$ is not p -stable. $\square$

3.3. The condition $e = f$. In the remainder of this paper, we consider two-term recurrence sequences for which $e = f$. These sequences have a more intricate structure and are less easy to handle than those for which $e > f$.

The two results in this section classify the stability of some of these sequences under the additional hypothesis that the discriminant D is not a quadratic residue modulo p . In particular, we identify one p^e -block whose sequences all fail to be p -stable and we show that those sequences that fail to be p -stable lie in a unique p -block.

THEOREM 3.4. *Suppose that $(\frac{D}{p}) = -1$ and $e = f$. Then there exists a p -regular recurrence $w'(a, b)$ that is not p -stable. Furthermore, we have the following stability criteria for $w(a, b) \in \mathcal{F}(a, b)$.*

- (a) *If $w(a, b)$ is a **mot** of $w'(a, b)$ modulo p^e , then $w(a, b)$ is not p -stable.*
- (b) *If $w(a, b)$ is not a **mot** of $w'(a, b)$ modulo p and also not a **mot** of $u(a, b)$ modulo p , then $w(a, b)$ is p -stable with $1 \leq \iota(p) \leq e$.*
- (c) *Suppose that $w(a, b)$ is not a **mot** of $w'(a, b)$ modulo p , but that $w(a, b)$ is a **mot** of $u(a, b)$ modulo p . Choose m maximal such that $m \leq e$ and $w(a, b)$ is a **mot** of $u(a, b)$ modulo p^m . Then $w(a, b)$ is p -stable with $1 \leq \iota(p) \leq e$.*

PROOF. Since $(\frac{D}{p}) = -1$, Theorem 2.10 implies that there is a recurrence $w'(a, b)$ that satisfies Hypothesis 2.9. Suppose that $r \geq 2e$. By the definition of r^* given in Section 2.8, $r^* = \lceil r/2 \rceil$, and $r - r^* = \lfloor r/2 \rfloor \geq (r - 1)/2$. Since $r > f$, Theorem 2.12(a) (with e in place of m) implies that there are at least s distinct p -regular residues d for which $\nu_w(d, p^r) \geq p^{r-r^*} \geq p^{(r-1)/2}$. In particular, $\max(\Omega_w(p^r))$ is unbounded as a function of r , and it follows that $w'(a, b)$ is not p -stable.

(a) Assume that $w(a, b)$ is in the same p^e -block as $w'(a, b)$. Then we can apply Theorem 2.12(a) (with e in place of m) in the same fashion as for $w'(a, b)$ itself, and it follows that $w(a, b)$ is not p -stable.

(b) Assume that $w(a, b)$ lies in a p -block different from those that contain $w'(a, b)$ and $u(a, b)$. As in the proof of Theorem 3.2, [7, Cor. 2.17] implies that $w(a, b)$ has no p -singular terms. But then, by Theorem 2.11, for all residues d ,

$$\nu(d, p^r) = \nu(d, p^f) \leq \nu(d, p) \quad (3.7)$$

when $r \geq f = e$. It follows that $w(a, b)$ is p -stable with $1 \leq \iota(p) \leq e$.

(c) Since $w(a, b)$ lies in a different p -block than $w'(a, b)$, Theorem 2.11 implies that for all p -regular residues d ,

$$v(d, p^r) = v(d, p^f) \leq v(d, p) \quad (3.8)$$

when $r \geq f = e$.

To handle the p -singular residues, we consider separately the cases that $m < e$ and $m = e$.

First, suppose that $m < e$. Clearly $m \geq 1$, so in this case we know that $e > 1$. Therefore, we can apply Theorem 2.15. Since $e = f$, it follows that $m > e - f$. As in the proof of Theorem 3.3(b), if d is p -singular, then

$$v(d, p^r) = p^{e-f} = 1 \quad (3.9)$$

when $r \geq e$. Thus, in this case, (3.8) and (3.9) imply that $w(a, b)$ is p -stable with $1 \leq \iota(p) \leq e$.

Now, suppose that $m = e$. Then, $w(a, b)$ is a **mot** of $u(a, b)$ modulo p^e and we apply Theorem 2.14. Suppose that $r \geq e$. Then, by the definitions of e^* and f^* given in Section 2.8, $e^* = e = f^*$, and hence, if d is p -singular, then

$$v(d, p^r) = \begin{cases} 0 & \text{if } p^e \nmid d, \\ s & \text{if } p^e \mid d. \end{cases} \quad (3.10)$$

In particular, $v(d, p^r)$ is independent of r . Now (3.8) and (3.10) imply that $w(a, b)$ is p -stable with $1 \leq \iota(p) \leq e$. $\square$

In Theorem 3.5, we identify families $\mathcal{F}(a, b)$ with the property that every p -regular sequence in $\mathcal{F}(a, b)$ fails to be p -stable.

THEOREM 3.5. *Suppose that $(\frac{D}{p}) = -1$, $e = 1$, and $h(p) = p + 1$. Then $(\frac{b}{p}) = -1$, and every p -regular recurrence $w(a, b) \in \mathcal{F}(a, b)$ is not p -stable.*

Furthermore, given any integer b' such that $(\frac{b'}{p}) = -1$, there exist integers a and b with $b \equiv b' \pmod{p}$ such that $(\frac{D}{p}) = -1$, $h(p) = p + 1$, and $e = 1$.

PROOF. Since $(\frac{D}{p}) = -1$ and $h(p) = p + 1$, [7, Thm. 2.14], which provides an explicit count of regular p -blocks, implies that there is only one regular p -block. Since $1 = e \geq f$, it follows that $e = f$. Consequently, Theorem 3.4 implies that this unique p -regular p -block contains a sequence that is not p -stable. Now, Theorem 3.4(a) implies that every p -regular sequence in $\mathcal{F}(a, b)$ fails to be p -stable. Finally, D. H. Lehmer [13, p. 441] has shown that if $(\frac{b}{p}) = 1$, then $h(p) \mid (p - (\frac{D}{p}))/2$. Since, by hypothesis, $h(p) = p + 1$, we conclude that $(\frac{b}{p}) = -1$.

Now, suppose that $(\frac{b}{p}) = -1$. By [19, Thm. 4], there exists a p -regular recurrence $u(a, b)$ such that $(\frac{D}{p}) = -1$ and $h(p) = p + 1$. If $e = 1$, we are done. Suppose instead that $e > 1$.

Let α and β be the characteristic roots of $u(a, b)$ and P a prime ideal lying over p in the algebraic number field $\mathbf{Q}(\alpha, \beta)$. Since $(\frac{D}{p}) = -1$, p is unramified. Moreover, the

characteristic polynomial is irreducible over $\mathbf{Q}(\alpha, \beta)/P$ and

$$\alpha - \beta \not\equiv 0 \pmod{P}. \quad (3.11)$$

Since the Frobenius automorphism exchanges the roots α and β , we also obtain

$$\begin{aligned} \alpha^p &\equiv \beta \pmod{P} & \text{and} & & p\alpha^p &\equiv p\beta \pmod{P^2}, \\ \beta^p &\equiv \alpha \pmod{P} & \text{and} & & p\beta^p &\equiv p\alpha \pmod{P^2}. \end{aligned} \quad (3.12)$$

Since $e \geq 1$, it follows that $h(p^2) = h(p) = p + 1$, and hence, by Lemma 2.4,

$$\alpha^{p+1} \equiv \beta^{p+1} \pmod{P^2}. \quad (3.13)$$

Now, consider the new sequence $u(a', b')$ with characteristic roots $\alpha' = \alpha + p$ and $\beta' = \beta + p$ and satisfying

$$\begin{aligned} a' &= \alpha' + \beta' = (\alpha + p) + (\beta + p) = \alpha + \beta + 2p = a + 2p \equiv a \pmod{p}, \\ b' &= \alpha' \beta' = (\alpha + p)(\beta + p) = \alpha\beta + (\alpha + \beta)p + p^2 = b + ap + p^2 \equiv b \pmod{p}. \end{aligned} \quad (3.14)$$

Since $a \equiv a' \pmod{p}$ and $b \equiv b' \pmod{p}$, we know that $h_{u(a', b')}(p) = p + 1$, and hence, by Lemma 2.4,

$$(\alpha + p)^{p+1} - (\beta + p)^{p+1} \equiv 0 \pmod{P}. \quad (3.15)$$

By the binomial theorem,

$$\begin{aligned} (\alpha + p)^{p+1} &\equiv \alpha^{p+1} + (p+1)p\alpha^p \equiv \alpha^{p+1} + p\alpha^p \pmod{P^2}, \\ (\beta + p)^{p+1} &\equiv \beta^{p+1} + (p+1)p\beta^p \equiv \beta^{p+1} + p\beta^p \pmod{P^2}. \end{aligned} \quad (3.16)$$

Thus, by (3.11), (3.12), and (3.13),

$$\begin{aligned} (\alpha + p)^{p+1} - (\beta + p)^{p+1} &\equiv (\alpha^{p+1} + p\alpha^p) - (\beta^{p+1} + p\beta^p) \pmod{P^2} \\ &\equiv p\alpha^p - p\beta^p \pmod{P^2} \\ &\equiv p\beta - p\alpha \pmod{P^2} \\ &\equiv p(\beta - \alpha) \pmod{P^2} \\ &\not\equiv 0 \pmod{P^2}. \end{aligned} \quad (3.17)$$

Consequently, $h_{u(a', b')}(p^2) > h_{u(a', b')}(p)$, and hence $e = 1$. It now follows that the sequence $u(a', b')$ satisfies the requirements of the theorem. $\square$

3.4. The condition $\text{ord}_{p^{2e}}(b) \mid p - 1$. In this section, we consider sequences for which $\text{ord}_{p^{2e}}(b) \mid p - 1$ and $p \nmid D$. Note that, by Theorems 2.10 and 2.13, these sequences satisfy $e = f$. Thus, the sequences here specialize the condition of the previous section; however, we replace the condition $(\frac{D}{p}) = -1$ with the less restrictive condition $p \nmid D$.

THEOREM 3.6. *Suppose that $p \nmid D$ and $\text{ord}_{p^{2e}}(b) \mid p - 1$. Then $v(a, b)$ is not p -stable. Furthermore, we have the following stability criteria for $w(a, b) \in \mathcal{F}(a, b)$.*

(a) *If $w(a, b)$ is a **mot** of $v(a, b)$ modulo p^e , then $w(a, b)$ is not p -stable.*

(b) If $w(a, b)$ is not a **mot** of $v(a, b)$ modulo p and not a **mot** of $u(a, b)$ modulo p , then $w(a, b)$ is p -stable with $1 \leq \iota(p) \leq e$.

(c) Suppose that $w(a, b)$ is not a **mot** of $v(a, b)$ modulo p , but that $w(a, b)$ is a **mot** of $u(a, b)$ modulo p . Choose m maximal such that $m \leq e$ and $w(a, b)$ is a **mot** of $u(a, b)$ modulo p^m . Then $w(a, b)$ is p -stable with $1 \leq \iota(p) \leq e$.

NOTE. In particular, if $p \nmid D$ and $b = \pm 1$, then each sequence $w(a, b) \in \mathcal{F}(a, b)$ satisfies the hypotheses of Theorem 3.6.

PROOF. (a) By Theorem 2.13, $v(a, b)$ satisfies Hypothesis 2.9 for $n = 0$. Suppose that $r > f$. Since $w(a, b)$ is in the same p^e -block as $v(a, b)$, Theorem 2.12(a) implies that there are at least s distinct p -regular residues d modulo p^r for which

$$\nu_w(d, p^r) \geq p^{r-r^*}. \quad (3.18)$$

Clearly, this implies that $\max(\Omega_w(p^r))$ is unbounded as a function of r , and hence $w(a, b)$ is not p -stable.

(b) As in the proof of Theorem 3.2(b), since $w(a, b)$ lies in a different p -block than $u(a, b)$, the elements of $w(a, b)$ are all p -regular. As in (a), Theorem 2.13 implies that $v(a, b)$ satisfies Hypothesis 2.9 for $n = 0$. Thus, Theorem 2.11 implies that the p -regular residues d modulo p^r satisfy

$$\nu(d, p^r) = \nu(d, p^f) \leq \nu(d, p) \quad (3.19)$$

when $r \geq f = e$. It follows that $w(a, b)$ is p -stable with $1 \leq \iota(p) \leq e$, as desired.

(c) As in (b), the p -regular residues d modulo p^r satisfy

$$\nu(d, p^r) = \nu(d, p^f) \leq \nu(d, p) \quad (3.20)$$

when $r \geq f = e$.

As in the proof of Theorem 3.4, to handle the p -singular residues we consider separately the cases that $m < e$ and $m = e$.

If $m < e$, we know that $e > 1$ and can apply Theorem 2.15. Since $e = f$, p -singular residues d satisfy

$$\nu(d, p^r) = p^{e-f} = 1 \quad (3.21)$$

when $r \geq e$. It follows that $w(a, b)$ is p -stable with $1 \leq \iota(p) \leq e$.

If $m = e$, we appeal to Theorem 2.14. Since $w(a, b)$ is a **mot** of $u(a, b)$ modulo p^e and $e = f$, Theorem 2.14 implies that p -singular residues d satisfy

$$\nu(d, p^r) = \begin{cases} 0 & \text{if } p^e \nmid d, \\ s & \text{if } p^e \mid d, \end{cases} \quad (3.22)$$

when $r \geq e$. In either case, the frequency is independent of r , and it follows that $w(a, b)$ is p -stable with $1 \leq \iota(p) \leq e$. $\square$

COROLLARY 3.7. Suppose that $p \nmid D$, that $\text{ord}_{p^{2e}}(b) \mid p - 1$, and that $(\frac{b}{p}) = 1$. Then $h(p) \mid (p - (\frac{D}{p}))/2$, and we have the following stability criteria for $w(a, b) \in \mathcal{F}(a, b)$.

(a) If $h(p)$ is odd and $w(a, b)$ is a **mot** of $u(a, b)$ modulo p^e , then $w(a, b)$ is p -stable with $1 \leq \iota(p) \leq e$.

(b) If $h(p)$ is even and $w(a, b)$ is a **mot** of $t(a, b)$ modulo p , then $w(a, b)$ is p -stable with $1 \leq \iota(p) \leq e$.

(c) If $h(p) = (p - (\frac{D}{p}))/2$ and $e = 1$, then $w(a, b)$ is not p -stable if and only if $w(a, b)$ is a **mot** of $v(a, b)$ modulo p .

(d) If $h(p) = (p - (\frac{D}{p}))/2$, $(p - (\frac{D}{p}))/2$ is odd, and $e = 1$, then $w(a, b)$ is p -stable if and only if $w(a, b)$ is a **mot** of $u(a, b)$ modulo p .

(e) If $h(p) = (p - (\frac{D}{p}))/2$, $(p - (\frac{D}{p}))/2$ is even, and $e = 1$, then $w(a, b)$ is p -stable if and only if $w(a, b)$ is a **mot** of $t(a, b)$ modulo p .

Conversely, if $\delta = \pm 1$ and b is any integer such that $\text{ord}_{p^{2e}}(b) \mid p - 1$ and $(\frac{b}{p}) = 1$, then there exists an integer a and a p -regular recurrence $w(a, b)$ such that $(\frac{D}{p}) = \delta$ and $h(p) = (p - (\frac{D}{p}))/2$.

PROOF. The fact that $h(p) \mid (p - (\frac{D}{p}))/2$ is proven in [13, p. 441].

(a) By Lemma 2.6, $w(a, b)$ is not a **mot** of $v(a, b)$ modulo p . Hence (a) follows from Theorem 3.6(c).

(b) We first note that, by definition, $t(a, b)$ is defined when p is odd, $(\frac{b}{p}) = 1$, and $h(p)$ is even. Moreover, $t(a, b)$ is not a **mot** of $u(a, b)$ or of $v(a, b)$. Therefore, (b) follows from Theorem 3.6(b).

(c), (d), (e) By [7, Thm. 2.14], the number of p -regular p -blocks in $\mathcal{F}(a, b)$ is

$$T_{\text{reg}}(p) = \frac{(p - (\frac{D}{p}))}{h(p)} = \frac{2h(p)}{h(p)} = 2. \quad (3.23)$$

One of these p -regular blocks contains the sequence $v(a, b)$. Since $e = 1$, Theorem 3.6 implies that $w(a, b)$ is not p -stable if and only if $w(a, b)$ lies in the same p -block as $v(a, b)$, and (c) follows immediately. If $h(p)$ is odd, then the other p -regular p -block contains $u(a, b)$, while if $h(p)$ is even, the other p -regular p -block contains $t(a, b)$. Thus (d) and (e) follow from (a) and (b), respectively.

To prove the partial converse, suppose that $\text{ord}_{p^{2e}}(b) \mid p - 1$, $(\frac{b}{p}) = 1$, and $\delta = \pm 1$. The existence of an integer a and corresponding regular second-order recurrence $w(a, b)$ such that $(\frac{D}{p}) = \delta$ and $h(p) = (p - (\frac{D}{p}))/2$ follows from [16, Thm. 12(i)] and [19, Thm. 4]. $\square$

3.5. The condition $b = \pm 1$. In this section, we sketch more detailed results in the case that $b = \pm 1$. These sequences have particular historical interest. Of course, the Fibonacci sequence itself belongs to the family $\mathcal{F}(1, -1)$. These are the sequences studied by Schinzel in the quintessential work [14], by Somer in [15, 17, 18, 20], and by Jacobson, Carroll, and Somer in [9].

In two theorems, dealing with $b = 1$ and $b = -1$ in turn, we describe the stability of sequences that belong to the same p^e -blocks as $u(a, b)$, $v(a, b)$, and $t(a, b)$. Since $b = \pm 1$, it is clear that $\text{ord}_{p^{2e}}(b) \mid p - 1$. Since we also assume that $p \nmid D$ in this section, the theorems here specialize those in the previous section. In particular, as in the previous section, each family $\mathcal{F}(a, b)$ studied here satisfies $e = f$.

THEOREM 3.8. Suppose that $b = 1$ and $p \nmid D$.

(a) If $h(p)$ is odd and $w(a, b)$ is a **mot** of $u(a, b)$ modulo p^e , then $w(a, b)$ is p -stable

and $\iota(p) = 1$. Furthermore, either $\lambda(p) \equiv 1 \pmod{2}$ or $\lambda(p) \equiv 2 \pmod{4}$, and, for all $r \geq 1$,

$$\Omega(p^r) = \begin{cases} \{0, 1\} & \text{if } \lambda(p) \equiv 1 \pmod{2}, \\ \{0, 2\} & \text{if } \lambda(p) \equiv 2 \pmod{4}. \end{cases} \quad (3.24)$$

(b) If $h(p)$ is even and $w(a, b)$ is a **mot** of $t(a, 1)$ modulo p^e , then $w(a, b)$ is p -stable and $\iota(p) = 1$. Furthermore, $\lambda(p) \equiv 0 \pmod{4}$ and $\Omega(p^r) = \{0, 2\}$ for all $r \geq 1$.

(c) If $w(a, b)$ is a **mot** of $v(a, b)$ modulo p^e , then $w(a, b)$ is not p -stable.

PROOF. (a) Since $w(a, b)$ is a **mot** of $u(a, b)$ modulo p^e , [7, Cor. 2.15] implies that $w(a, b)$ is a **mot** of $u(a, b)$ modulo p^r for all $r \geq e$. Therefore, $w(a, b)$ is a **mot** of $u(a, b)$ modulo p^r for all $r \geq 1$. Since two sequences in the same p^r -block have the same residue frequencies, we may assume that $w(a, b) = u(a, b)$.

By hypothesis, $h(p)$ is odd, so Lemma 2.6 implies that $w(a, b)$ is not a **mot** of $v(a, b)$ modulo p . Thus, by Theorem 3.6(c), $w(a, b)$ is p -stable with $1 \leq \iota(p) \leq e$.

From [15, Thm. 16], we see that $\lambda(p) \equiv 1 \pmod{2}$ or $\lambda(p) \equiv 2 \pmod{4}$ and

$$s = \begin{cases} 2 & \text{when } \lambda(p) \equiv 1 \pmod{2}, \\ 1 & \text{when } \lambda(p) \equiv 2 \pmod{4}. \end{cases} \quad (3.25)$$

In the case that $\lambda(p) \equiv 1 \pmod{2}$, [18, Thm. 4] shows that $\Omega(p) = \{0, 1\}$. Since, as previously observed, $e = f$, Theorem 2.14 implies that if $r \geq e$, then the p -singular residues d satisfy

$$\nu(d, p^r) = \begin{cases} 0 & \text{if } p^e \nmid d, \\ s = 1 & \text{if } p^e \mid d. \end{cases} \quad (3.26)$$

On the other hand, by Theorem 2.11, if $r \geq e$, then the p -regular residues d satisfy

$$\nu(d, p^r) = \nu(d, p^f) \leq \nu(d, p). \quad (3.27)$$

Clearly, (3.26) and (3.27) imply that $\Omega(p^r) = \{0, 1\}$ when $r \geq e = f$. On the other hand, if $r \leq f$, then $\lambda(p^r) = \lambda(p^f)$ and it is clear that $\nu(d, p) \geq \nu(d, p^r)$. It follows that $\Omega(p^r) = \{0, 1\}$ for all $r \geq 1$. In particular, $\iota(p) = 1$.

In the case that $\lambda(p) \equiv 2 \pmod{4}$, [18, Thm. 5] shows that $\Omega_u(p) = \{0, 2\}$. As before, Theorem 2.14 implies that if $r \geq e$, then the p -singular residues d satisfy

$$\nu(d, p^r) = \begin{cases} 0 & \text{if } p^e \nmid d, \\ s = 2 & \text{if } p^e \mid d. \end{cases} \quad (3.28)$$

On the other hand, the p -regular residues d continue to satisfy (3.27). Moreover, the same symmetry argument used to prove [18, Thm. 5] shows that 1 cannot occur as $\nu(d, p^r)$ for a p -regular residue d . It now follows from (3.28) and (3.27) that $\Omega(p^r) = \{0, 2\}$ when $r \geq e$, and, as in the previous paragraph, $\Omega(p^r) = \{0, 2\}$ for all $r \geq 1$. Once again, we also conclude that $\iota(p) = 1$.

(b) Since $w(a, b)$ is a **mot** of $t(a, b)$ modulo p^e , [7, Cor. 2.15] implies that $w(a, b)$ is a **mot** of $t(a, b)$ modulo p^r for all $r \geq e$. Therefore $w(a, b)$ is a **mot** of $t(a, b)$ modulo

p^r for all $r \geq 1$. Since two sequences in the same p^r -block have the same residue frequencies, we may assume that $w(a, b) = t(a, b)$.

By hypothesis, $h(p)$ is even and $w(a, b)$ is a **mot** of $t(a, b)$ modulo p . Consequently, Corollary 3.7(b) implies that $w(a, b)$ is p -stable with $1 \leq \iota(p) \leq e$.

By [18, Thm. 3(ii)], $\lambda(p) \equiv 0 \pmod{4}$. By using the technique of [18, Thms. 4-6] together with the symmetry properties of $t(a, b)$ given in [20, Lem. 5], it is easy to see that $s = 2$ for this sequence, $\Omega(p) = \{0, 2\}$, and that 1 cannot occur as $v(d, p^r)$ for a p -regular residue d . The argument can now be completed as in (a).

(c) This follows immediately from Theorem 3.6(a). $\square$

THEOREM 3.9. *Suppose that $b = -1$ and $p \nmid D$.*

(a) *If $h(p)$ is odd and $w(a, b)$ is a **mot** of $u(a, b)$ modulo p^e , then $w(a, b)$ is p -stable. Furthermore, $p \equiv 1 \pmod{4}$ and*

- (1) *if $p = 5$ and $e = 1$, then $\iota(p) = 1$, and $\Omega(p^r) = \{2, 4\}$ for all $r \geq 1$;*
- (2) *if $p = 5$ and $e > 1$, then $\iota(p) = 2$, and $\Omega(p) = \{2, 4\}$ and $\Omega(p^r) = \{0, 2, 4\}$ for all $r \geq 2$;*
- (3) *if $p > 5$, then $\iota(p) = 1$, and $\Omega(p^r) = \{0, 2, 4\}$ for all $r \geq 1$.*

(b) *If $h(p)$ is even, $p \equiv 1 \pmod{4}$, and $w(a, b)$ is a **mot** of $t(a, b)$ modulo p , then $w(a, b)$ is p -stable and $1 \leq \iota(p) \leq e$. Furthermore, $\Omega(p^r) = \{0, 1\}$, $\{0, 1, 2\}$, or $\{0, 2\}$ for all $r \geq 1$.*

(c) *If $w(a, b)$ is a **mot** of $v(a, b)$ modulo p^e , then $w(a, b)$ is not p -stable.*

PROOF. (a) Since $w(a, b)$ is a **mot** of $u(a, b)$ modulo p^e and $u(a, b)$ is p -regular, [7, Cor. 2.15] implies that $w(a, b)$ is a **mot** of $u(a, b)$ for all $r \geq e$. It follows that $w(a, b)$ is a **mot** of $u(a, b)$ for all $r \geq 1$, and we may assume that $w(a, b) = u(a, b)$.

By [23, Thm. 4], $h(p^r)$ is odd if and only if both $\lambda(p^r) \equiv 4 \pmod{8}$ and $E(p^r) = 4$. In particular, since $h(p)$ is odd, $s = 4$. Moreover, [15, Lem. 3] implies that $p \equiv 1 \pmod{4}$. Now, by Euler's criterion, $\left(\frac{-1}{p}\right) = 1$, and we can apply Corollary 3.7(a) to conclude that $w(a, b)$ is p -stable with $1 \leq \iota(p) \leq e$. If $r \geq 1$, the same methods used to prove [17, Thm. 9] can be used to show that $v(d, p) = 2$ or $v(d, p) = 4$ when $v(d, p^r) \neq 0$.

Now, suppose that $p = 5$ and $e = 1$. Then $\iota(5) = 1$, and an explicit computation shows that $h(5)$ is odd if and only if $a \equiv 2 \pmod{5}$ or $a \equiv 3 \pmod{5}$. In both cases $\lambda(5) = 12$ and $\Omega(5) = \{2, 4\}$.

Next, suppose that $p = 5$ and $e > 1$, and let $e^* = \min(r, e)$. By Theorem 2.14, if d is p -singular, then, for all r ,

$$v(d, p^r) = \begin{cases} 0 & \text{if } p^{e^*} \nmid d, \\ s = 4 & \text{if } p^{e^*} \mid d. \end{cases} \quad (3.29)$$

In particular, when $r \geq 2$, we obtain $v(p, p^r) = 0$ and $v(0, p^r) = 4$.

Since, by Lemma 2.6, $u(a, b)$ is not a **mot** of $v(a, b)$, we can also apply Theorem 2.11. Thus, for p -regular residues d ,

$$v(d, p^r) = v(d, p^f) \leq v(d, p) \quad (3.30)$$

when $r \geq f = e$. Since $v(1, 5) = 2$, it follows that $2 \in \Omega(p^r)$ for all $r \geq 1$. Now, $\Omega(p^r) = \{0, 2, 4\}$ when $r \geq 2$. Since $\Omega(5) = \{2, 4\}$ whenever $h(5)$ is odd, we conclude that $\iota(p) = 2$.

Finally, suppose that $p > 5$. Since $p \equiv 1 \pmod{4}$, we know that $p > 7$, and the result is proven in [9].

(b) As in (a), we may assume that $w(a, b) = t(a, b)$. Since $p \equiv 1 \pmod{4}$, Euler's criterion implies that $(\frac{-1}{p}) = 1$. Hence, by Corollary 3.7(b), $w(a, b)$ is stable, with $1 \leq \iota(p) \leq e$. Using the symmetry properties for $t(a, b)$ modulo p given in [20, Lem. 5] and employing methods similar to those used in the proofs of [17, Thms. 5, 7, and 9], we can show that $\Omega(p) = \{0, 1\}$, $\{0, 1, 2\}$, or $\{0, 2\}$. Finally, if $r \leq f = e$, then $\nu(d, p) \geq \nu(d, p^r)$. It follows that $\Omega(p^r) = \{0, 1\}$, $\{0, 1, 2\}$, or $\{0, 2\}$ for all $r \geq 1$.

(c) This follows immediately from Theorem 3.6(a). $\square$

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