DYNAMICS OF A CERTAIN SEQUENCE OF POWERS

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Abstract. For any nonzero complex number \( z \) we define a sequence \( a_1(z) = z, \ldots, a_{n+1}(z) = z^{a_n(z)}, \) \( n \in \mathbb{N} \). We attempt to describe the set of these \( z \) for which the sequence \( \{a_n(z)\} \) is convergent. While it is almost impossible to characterize this convergence set in the complex plane \( \mathbb{C} \), we achieved it for positive reals. We also discussed some connection to the Euler's functional equation.

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1. Introduction. In this paper, for any \( z \in \mathbb{C} \setminus \{0\} \), we consider the sequence

\[
a_1(z) = z, \quad a_2(z) = z^2 = z^{a_1(z)}, \ldots, a_{n+1}(z) = z^{a_n(z)}, \quad n = 1, 2, \ldots. \quad (1.1)
\]

We wish to characterize the convergence set \( L := \{z \in \mathbb{C} \setminus \{0\} : \{a_n(z)\} \text{ is convergent}\} \). Since \( 1 \in L \), this set is nonempty. For a complex value of \( z \), the mapping \( z \to a_n(z) \) is a multivalued operation and \( a_n(z) \) usually represents an infinite set. If we agree to use the main branch of the complex logarithm (as it is done, e.g., in MATLAB), the above exponentiation becomes a single-valued operation. For arbitrary \( z \), it is very easy to run out of accuracy of any available computational software in a few iterations. Thus, it is practically impossible to describe the convergence set \( L \). Also, one can show that the sequence \( \{a_n(z)\} \) cannot be obtained as a functional iteration, i.e., there exists no function \( f : \mathbb{C} \to \mathbb{C} \) such that \( a_n(z) = f^{[n]}(z) = f \circ f \circ \ldots \circ f(z) \) (\( n \)-times). This sequence has nasty numerical features. Nevertheless, the situation is not hopeless when it comes to considering real values of \( a_n(z) \).

2. Main result. Our main goal is to demonstrate the following.

**Theorem 2.1.** We have \( L \cap \mathbb{R}_+ = [e^{-e}, e^{1/e}] \).

Before we proceed with the proof of Theorem 2.1, the following lemma is in order.

**Lemma 2.2.** For the function \( f(t) = \ln(t)/t \) defined on \( \mathbb{R}_+ = (0, +\infty) \), the following hold true:

(a) range \( f \) = \( (-\infty, 1/e] \),

(b) for any \( 0 < c < 1/e \), the equation \( f(t) = c \) has exactly two solutions, while for \( c = 1/e \) exactly one, namely \( t^* = e \).

**Proof.** The first derivative of the function \( f(t) \) is \( f'(t) = (1 - \ln(t))/t^2 \), so that \( f'(t) = 0 \) if and only if \( t = e \), \( f \) is increasing on \( (0, e] \) and decreasing on \( [e, +\infty) \). Since
lim_{t \to 0^+} f(t) = -\infty \quad \text{and} \quad \lim_{t \to +\infty} f(t) = 0, \text{ which implies statement (a). Since} \quad t^* = e \quad \text{is the only maximizer of} \quad f(t) \quad \text{in} \quad (0, +\infty), \text{ (b) follows from the intermediate value theorem and the fact that} \quad \lim_{t \to 0^+} f(t) = 0. \text{ The proof is complete.} \quad \square

Figure 2.1.

Note that if \( t_c^{(1)} \) and \( t_c^{(2)} \) denote two distinct solutions of the equation \( f(t) = c \), then \( t_c^{(1)} \to 1^+ \) and \( t_c^{(2)} \to +\infty \) as \( c \to 0^+ \). This remark follows from the piecewise monotonicity of the function \( f(t) \) and the fact that \( \lim_{t \to 0^+} f(t) = 0 \) and \( \lim_{t \to +\infty} f(t) = 0 \). By putting \( c = \ln(x) \) in Lemma 2.2, we get the following.

**Corollary 2.3.** For any given \( x \in (1, e^{1/e}) \) there exist exactly two solutions of the equation \( x^\alpha = x \), while for \( x = e^{1/e} \), exactly one.

**Lemma 2.4.** Let \( K(x) := \{0 \leq t \leq 1 : h_x(t) = 0\} \), where \( h_x(t) = x^{x^t} - t \) and \( x > 0 \). The following statements hold true:

(a) For \( x \in (0, e^{-e}) \), \( K(x) \) consists of three distinct elements.
(b) For \( x \in [e^{-e}, 1) \), \( K(x) \) is a singleton.

**Proof.** Instead of giving a rigorous analytic proof similar to Bomberger [1], we rather illustrate our point with an aid of appropriate graphs.

**Case 1** (\( x \in (0, e^{-e}) \)). See Figure 2.2.

**Case 2** (\( x = e^{-e} \)). See Figure 2.3.

In fact, \( h'_x(t) = x^t x^{x^t} (\ln(x))^2 - 1 \), so \( h'_e^{-e}(1/e) = 0 \). Also, \( h_{e^{-e}}(1/e) = 0 \).

**Case 3** (\( x > e^{-e} \)). See Figure 2.4.

This completes our proof.

**Proof of Theorem 2.1**

**Case 1** \( (0 < x < e^{-e}) \). By Lemma 2.4 (see Figure 2.2), there exist numbers \( 0 < \alpha_1 < \)
\[ \alpha_2 < \alpha_3 < 1 \text{ such that} \]
\[ x^{\alpha_i^2} = \alpha_i \quad \text{for } i = 1, 2, 3. \] (2.1)

While \( \alpha_2 \) is the fixed point of the function \( g_x(t) = x^t \), \( \alpha_1 \) and \( \alpha_3 \) are points of period 2 (\( x^{\alpha_1^2} = \alpha_2 \) and \( x^{\alpha_3^2} = \alpha_1 \)).

Generating the sequence \( \{a_n(x)\} \) by putting \( a_1(x) := x = g_x(1) \) and \( a_{n+1}(x) = g_x(a_n(x)) \) for \( n \geq 1 \), as illustrated by Figure 2.5, we get
\[ a_1(x) < a_3(x) < \cdots < a_{2n-1}(x) < \alpha_1, \quad a_2(x) > a_4(x) > \cdots > a_{2n}(x) > \alpha_3. \] (2.2)

Thus, \( \{a_n(x)\} \) diverges but consists of two complementary convergent subsequences.

**Case 2** (\( x = e^{-e} \)). In this case, \( \{a_n(x)\} \) is convergent and the limit is \( 1/e \). Indeed, the sequence \( \{a_n(x)\} \) consists of two complementary subsequences and each of them is convergent to a point of period 2. But in this case, \( \alpha_1 = \alpha_2 = \alpha_3 = 1/e \), and the two points of period 2 collapse to the fixed point of the function \( g_{e^{-e}}(t) \). Hence, \( \{a_n(x)\} \) converges to \( 1/e \). However, its convergence is very slow (e.g., \( a_{1,000,000}(e^{-e}) \approx 0.36697888108297 \) and \( |a_{1,000,000}(e^{-e}) - 1/e| \approx 0.00180 \).

**Case 3** (\( e^{-e} < x \leq 1 \)). For \( x < 1 \) we reason the same way as in Case 2, so the sequence \( \{a_n(x)\} \) converges to a unique fixed point of the function \( g_x(t) \). For \( x = 1 \), the statement is trivial.

**Case 4** (\( 1 < x < e^{1/e} \)). If \( \{a_n(x)\} \) is convergent to a limit \( \alpha \), \( \{a_n(x)\} \) also converges to \( \alpha \), moreover, \( a_{n+1}(x) = x^{a_n(x)} \) converges to \( x^\alpha \). Consequently, \( x^\alpha = \alpha \). By Corollary 2.3, there are exactly two fixed points \( \alpha_1 \) and \( \alpha_2 \) (\( \alpha_1 < \alpha_2 \)) of the function \( g_x(t) \). Obviously, \( 1 < \alpha_1 < e < \alpha_2 \). Since \( g_x(t) \) is increasing, \( 1 < x < \alpha_1 \), and we check by the math induction that \( a_1(x) < a_2(x) < \cdots < a_n(x) < a_1 \), so the limit of the sequence \( \{a_n(x)\} \) exists and is just a fixed point \( \alpha_1 \).

![Figure 2.2](image-url)
Case 5 \((x = e^{1/e})\). As before, \(1 < x < e\) and \(\{a_n(x)\}\) is increasing and bounded by \(e\), thus convergent to the only fixed point of \(g_{e^{1/e}}(t)\), namely \(e\) itself.

Case 6 \((x > e^{1/e})\). By Corollary 2.3, no fixed point of \(g_x(t)\) exists, so the sequence \(\{a_n(x)\}\) does not converge. Now, the proof is complete.

We note that in Case 4 of the above proof, \(\alpha_1\) is an attracting fixed point, while \(\alpha_2\) is repelling fixed point for \(g_x(t)\). In the limiting Case 5, the point \(\alpha = e\) is left
attracting and right repelling. Our Corollary 2.3 and Lemma 2.4 have some interesting relations to certain functional equations, one of them considered already by Euler (see Sierpiński [2]):

(a) $0 < x < e^{-e}$. As we have already noticed, there exist exactly three zeros of the function $h_x(t)$, say, $\alpha_1 < \alpha_2 < \alpha_3$, so that $x^{\alpha_i} = \alpha_i$ for $i = 1, 2, 3$, see (2.1); $\alpha_1$ and $\alpha_3$ are points of period 2 of the function $g_x(t)$, i.e., $x^{\alpha_1} = \alpha_3$ and $x^{\alpha_3} = \alpha_1$; $(\alpha_2$ is just the fixed point of the function $g_x(t)$). Thus, 

$$\alpha_1^{\alpha_1} = (x^{\alpha_3})^{\alpha_1} = (x^{\alpha_1})^{\alpha_3} = \alpha_3^{\alpha_3}. \quad (2.3)$$

In other words, $\alpha_1$ and $\alpha_3$ satisfy the functional equation $u^u = v^v$. For a detailed account on solving this equation we refer to Bomberger [1]. Observe that all $\alpha_i$'s are in $(0,1)$.

(b) $1 < x < e^{1/e}$. As we noted after Lemma 2.2, there exist two distinct fixed points $\alpha_1$ and $\alpha_2$ of $g_x(t)$, and when $x \to 1^+$, $\alpha_1 \to 1^+$, and $\alpha_2 \to +\infty$. Since $x^{\alpha_1} = \alpha_1$ and $x^{\alpha_2} = \alpha_2$, we get 

$$\alpha_1^{\alpha_1} = (x^{\alpha_2})^{\alpha_1} = (x^{\alpha_1})^{\alpha_1} = \alpha_2^{\alpha_1}, \quad (2.4)$$

thus, $\alpha_1$ and $\alpha_2$ are related by the equation $u^u = v^u$ ($u, v > 0$), called the Euler equation.

(c) $e^{-e} \leq x \leq 1$. Now, $\alpha_1$ and $\alpha_3$ considered in (a) collapse to the same point and both types of equations considered in (a) and (b) are automatically satisfied.

![Figure 2.5](image-url)
For the sake of completeness of this exposition we describe the set of solutions of the Euler equation.

**Proposition 2.5.** The set of solutions of the Euler equation \( u^v = v^u \), where \( u \neq v \), \( u, v > 0 \) is given by a parametrized family \( \{(u, v) = (\alpha^{1/(\alpha-1)}, \alpha \alpha/(\alpha-1)): \alpha > 1\} \).

**Proof.** We note that if \( 0 < u < v \leq 1 \), then \( u < u^v < u^u < v^u \), and a pair \((u, v)\) cannot be a solution. Similarly, when \( 0 < u \leq 1 < v \). Observe that \( \ln(u)/u = \ln(v)/v \), and if this common value is less than \( 1/e \), by Lemma 2.2 there exist exactly two solutions \( u \) and \( v \) of this equation and \( 1 < u < e < v \). We postulate a solution in the form \( u = t \) and \( v = t^\alpha \), where \( t > 1 \) and \( \alpha > 0 \) are parameters. Thus,

\[
t^\alpha = (t^\alpha)^t = t^\alpha, \tag{2.5}
\]

so \( t^\alpha = \alpha t \) and, in consequence, \( t = \alpha^{1/(\alpha-1)} \). Hence, \( (u, v) = (\alpha^{1/(\alpha-1)}, \alpha \alpha/(\alpha-1)) \). For any \( 1 < \alpha > 0 \), \( \alpha^{1/(\alpha-1)} > 1 \), so in order to keep \( u < v \), \( \alpha \) must be greater than 1. We check that the pair \((u, v)\) actually solves the equation:

\[
u^v = (\alpha^{1/(\alpha-1)})^{\alpha^{\alpha/(\alpha-1)}} = \alpha^{1/(\alpha-1)} \alpha^{\alpha/(\alpha-1)} = \alpha^{1/(\alpha-1)} \alpha^{1/(\alpha-1)} = v^u. \tag{2.6}
\]

Now, we show that this parametrized family exhausts all possible solutions of the Euler equation such that \( u < v \). In fact, if \( \alpha \to 1^+ \), \( \alpha^{1/(\alpha-1)} \to e \), and \( \alpha^{\alpha/(\alpha-1)} \to e \); if \( \alpha \to +\infty \), \( \alpha^{1/(\alpha-1)} \to 1 \), and \( \alpha^{\alpha/(\alpha-1)} \to +\infty \). The result now follows from the intermediate value theorem.

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