THE PRODUCT OF r^{-k} AND $\nabla\delta$ ON $\mathbb{R}^m$

C. K. LI

(Received 21 December 1999 and in revised form 4 January 2000)

ABSTRACT. In the theory of distributions, there is a general lack of definitions for products and powers of distributions. In physics (Gasiorowicz (1967), page 141), one finds the need to evaluate δ^2 when calculating the transition rates of certain particle interactions and using some products such as $(1/x) \cdot \delta$. In 1990, Li and Fisher introduced a “computable” delta sequence in an m -dimensional space to obtain a noncommutative neutrix product of r^{-k} and $\Delta\delta$ (Δ denotes the Laplacian) for any positive integer k between 1 and $m - 1$ inclusive. Cheng and Li (1991) utilized a net $\delta_\epsilon(x)$ (similar to the $\delta_n(x)$) and the normalization procedure of $\mu(x)x_+^\lambda$ to deduce a commutative neutrix product of r^{-k} and δ for any positive real number k . The object of this paper is to apply Pizetti’s formula and the normalization procedure to derive the product of r^{-k} and $\nabla\delta$ (∇ is the gradient operator) on $\mathbb{R}^m$. The nice properties of the δ -sequence are fully shown and used in the proof of our theorem.

Keywords and phrases. Pizetti’s formula, delta sequence, neutrix limit and distribution.

2000 Mathematics Subject Classification. Primary 46F10.

1. Introduction. Let $\rho(x)$ be a fixed infinitely differentiable function with the following properties:

- (i) $\rho(x) \geq 0$,
- (ii) $\rho(x) = 0$ for $|x| \geq 1$,
- (iii) $\rho(x) = \rho(-x)$,
- (iv) $\int_{-1}^1 \rho(x) dx = 1$.

The function $\delta_n(x)$ is defined by $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \dots$. It follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

Now let $\mathcal{D}$ be the space of infinitely differentiable functions of a single variable with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. Then if f is an arbitrary distribution in $\mathcal{D}'$, we define

$$f_n(x) = (f * \delta_n)(x) = (f(t), \delta_n(x - t)) \quad (1.1)$$

for $n = 1, 2, \dots$. It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x)$ in $\mathcal{D}'$.

The following definition for the noncommutative neutrix product $f \cdot g$ of two distributions f and g in $\mathcal{D}'$ was given by Fisher in [2].

DEFINITION 1.1. Let f and g be distributions in $\mathcal{D}'$ and let $g_n = g * \delta_n$. We say that the neutrix product $f \cdot g$ of f and g exists and is equal to h if

$$N - \lim_{n \rightarrow \infty} (f g_n, \phi) = (h, \phi) \quad (1.2)$$

for all functions ϕ in $\mathcal{D}$, where N is the neutrix (see [6]) having domain $N' = \{1, 2, \dots\}$ and range N'' the real numbers, with negligible functions that are finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \ln^r n \quad (\lambda > 0, r = 1, 2, \dots) \quad (1.3)$$

and all functions of n which converge to zero in the normal sense as n tends to infinity.

The product of Definition 1.1 is not symmetric and hence $f \cdot g \neq g \cdot f$ in general.

Extending definitions of products from one-dimensional space $\mathbb{R}$ to m -dimensional space $\mathbb{R}^m$ by using appropriate delta-sequences has recently been an interesting topic in distribution theory. In order to define a neutrix product of two separable forms of distributions in $\mathcal{D}'_m$ (an m -dimensional space of distributions), Fisher and Li provided the following definition in [3].

DEFINITION 1.2. Let $f(x)$ and $g(x)$ be distributions in $\mathcal{D}'_m$, where $x = (x_1, x_2, \dots, x_m)$. The function $g_n(x)$ is defined by

$$g_n(x) = (g * \delta_n)(x), \quad (1.4)$$

where $\delta_n(x) = \delta_{n_1}(x_1) \cdots \delta_{n_m}(x_m) = n_1 \rho(n_1 x_1) \cdots n_m \rho(n_m x_m)$. Hence $\{\delta_n(x)\}$ is a regular sequence converging to the Dirac delta-function $\delta(x)$. The neutrix product $f \cdot g$ is defined to be equal to h if

$$N - \lim_{n_1 \rightarrow \infty} \cdots N - \lim_{n_m \rightarrow \infty} (f g_n, \phi) = (h, \phi) \quad (1.5)$$

for all ϕ in $\mathcal{D}_m$ (an m -dimensional Schwartz space).

With Definition 1.2, Fisher and Li (also in [3]) show the following results.

Let

$$x^r = x_1^{-r_1} \cdots x_m^{-r_m} \quad \text{and} \quad \delta^{(p)}(x) = \delta^{(p_1)}(x_1) \cdots \delta^{(p_m)}(x_m). \quad (1.6)$$

Then the noncommutative neutrix product $x^{-r} \cdot \delta^{(p)}(x)$ exists and

$$x^{-r} \cdot \delta^{(p)}(x) = \frac{(-1)^r p!}{(p+r)!} \delta^{(p+r)}(x) \quad (1.7)$$

for $r_1, \dots, r_m = 1, 2, \dots$ and $p_1, \dots, p_m = 0, 1, 2, \dots$.

The following work on the commutative neutrix product of distributions on $\mathbb{R}^m$ is due to Cheng and Li (see [1]).

Let $\mathbb{R}^m$ be an Euclidean space with dimension m , and let $\rho(s)$, for $s \in \mathbb{R}$, be a fixed infinitely differentiable function having the properties:

- (i) $\rho(s) \geq 0$,
- (ii) $\rho(s) = 0$ for $|s| \geq 1$,
- (iii) $\rho(s) = \rho(-s)$,

$$(iv) \int_{|x| \leq 1} \rho(|x|^2) dx = 1, \quad x \in \mathbb{R}^m.$$

The property (iv) in the spherical coordinates is represented as

$$(v) \Omega_m \int_0^1 \rho(s^2) s^{m-1} ds = 1,$$

where Ω_m is the surface area of the unit sphere in $\mathbb{R}^m$. Putting $\delta_\epsilon(x) = \epsilon^{-m} \rho(|\epsilon^{-1}x|^2)$, where $\epsilon > 0$, it follows that ϵ -net $\{\delta_\epsilon(x)\}$ converges to the Dirac delta-function $\delta(x)$.

DEFINITION 1.3. Let f and g be arbitrary distributions in $\mathcal{D}'_m$ and let

$$f_\epsilon = f * \delta_\epsilon, \quad g_\epsilon = g * \delta_\epsilon. \quad (1.8)$$

We say that the neutrix product $f \cdot g$ of f and g exists and is equal to h on the open domain $\Omega \subseteq \mathbb{R}^m$ if the neutrix limit

$$N - \lim_{\epsilon \rightarrow 0^+} \frac{1}{2} \{(f \cdot g_\epsilon, \phi) + (g \cdot f_\epsilon, \phi)\} = (h, \phi) \quad (1.9)$$

for all test functions ϕ with compact support contained in the domain Ω , where N is the neutrix having domain $N' = \mathbb{R}^+$, the positive numbers, and range $N'' = \mathbb{R}$, the real numbers, with negligible functions that are linear sums of the functions

$$\epsilon^{-\lambda} \ln^{r-1} \epsilon, \quad \ln^r \epsilon \quad (1.10)$$

for $\lambda > 0$ and $r = 1, 2, \dots$, and all functions of ϵ which converge to zero as ϵ tends to zero.

In this paper, we would like to give a definition for the noncommutative neutrix product $f \cdot g$ of two distributions f and g in $\mathcal{D}'_m$ by applying the below δ -sequence. This definition is particularly useful in computing products of distributions of the variable r (radius).

From now on we let $\rho(s)$ be a fixed infinitely differentiable function defined on $\mathbb{R}^+ = [0, \infty)$ having the properties:

- (i) $\rho(s) \geq 0$,
- (ii) $\rho(s) = 0$ for $s \geq 1$,
- (iii) $\int_{\mathbb{R}^m} \delta_n(x) dx = 1$,

where $\delta_n(x) = C_m n^m \rho(n^2 r^2)$ and C_m is the constant satisfying (iii).

It follows that $\{\delta_n(x)\}$ is a regular δ -sequence of infinitely differentiable functions converging to $\delta(x)$ because of the above three conditions. The following definition will be applied in Section 3 to evaluate our product mentioned in the abstract.

DEFINITION 1.4. Let f and g be distributions in $\mathcal{D}'(m)$ and let

$$g_n(x) = (g * \delta_n)(x) = (g(x-t), \delta_n(t)), \quad (1.11)$$

where $t = (t_1, t_2, \dots, t_m)$. We say that the neutrix product $f \cdot g$ of f and g exists and is equal to h if

$$N - \lim_{n \rightarrow \infty} (f g_n, \phi) = (h, \phi), \quad (1.12)$$

where $\phi \in \mathcal{D}_m$.

2. The distribution r^λ . Let $r = (x_1^2 + \cdots + x_m^2)^{1/2}$ and consider the functional r^λ (see [5]) defined by

$$(r^\lambda, \phi) = \int_{\mathbb{R}^m} r^\lambda \phi(x) dx, \quad (2.1)$$

where $\operatorname{Re} \lambda > -m$ and $\phi(x) \in \mathcal{D}_m$. Because the derivative

$$\frac{\partial}{\partial \lambda} (r^\lambda, \phi) = \int r^\lambda \ln r \phi(x) dx \quad (2.2)$$

exists, the functional r^λ is an analytic function of λ for $\operatorname{Re} \lambda > -m$.

For $\operatorname{Re} \lambda \leq -m$, we should use the following identity (2.4) to define its analytic continuation. For $\operatorname{Re} \lambda > 0$, we could deduce

$$\Delta(r^{\lambda+2}) = (\lambda+2)(\lambda+m)r^\lambda \quad (2.3)$$

simply by calculating the left-hand side, where Δ is the Laplacian operator. By iteration we find, for any integer k , that

$$r^\lambda = \frac{\Delta^k r^{\lambda+2k}}{(\lambda+2) \cdots (\lambda+2k)(\lambda+m) \cdots (\lambda+m+2k-2)}. \quad (2.4)$$

On making substitution of spherical coordinates in (2.1), we come to

$$(r^\lambda, \phi) = \int_0^\infty r^\lambda \left\{ \int_{r=1} \phi(r\omega) d\omega \right\} r^{m-1} dr, \quad (2.5)$$

where $d\omega$ is the hypersurface element on the unit sphere. The integral appearing in the above integrand can be written in the form

$$\int_{r=1} \phi(r\omega) d\omega = \Omega_m S_\phi(r), \quad (2.6)$$

where Ω_m is the hypersurface area of the unit sphere imbedded in Euclidean space of m dimensions, and $S_\phi(r)$ is the mean value of ϕ on the sphere of radius r .

It was proved in [5] that $S_\phi(r)$ is infinitely differentiable for $r \geq 0$, has bounded support, and that

$$S_\phi(r) = \phi(0) + a_1 r^2 + a_2 r^4 + \cdots + a_k r^{2k} + o(r^{2k}) \quad (2.7)$$

for any positive integer k . From (2.5) and (2.6), we obtain

$$(r^\lambda, \phi) = \Omega_m \int_0^\infty r^{\lambda+m-1} S_\phi(r) dr \quad (2.8)$$

which indicates the application of $\Omega_m x_+^\mu$ with $\mu = \lambda + m - 1$ to the testing function $S_\phi(r)$. Using the following Laurent series for x_+^λ about $\lambda = -k$,

$$x_+^\lambda = \frac{(-1)^{k-1} \delta^{(k-1)}(x)}{(k-1)!(\lambda+k)} + x_+^{-k} + (\lambda+k)x_+^{-k} \ln x + \cdots \quad (2.9)$$

we could show that the residue of $(r^\lambda, \phi(x))$ at $\lambda = -m - 2k$ for nonnegative integer k is given by

$$\Omega_m \frac{(\delta^{(2k)}, \phi(x))}{(2k)!} = \Omega_m \frac{S_\phi^{(2k)}(0)}{(2k)!}. \quad (2.10)$$

On the other hand, the residue of the function r^λ of (2.4) for the same value of λ is

$$\frac{\Omega_m \Delta^k \delta(x)}{2^k k! m(m+2) \cdots (m+2k-2)}. \quad (2.11)$$

(See [5].) Therefore we get

$$S_\phi^{(2k)}(0) = \frac{(2k)! \Delta^k \phi(0)}{2^k k! m(m+2) \cdots (m+2k-2)}. \quad (2.12)$$

This result can be used to write out the Taylor's series for $S_\phi(r)$, namely

$$\begin{aligned} S_\phi(r) &= \phi(0) + \frac{1}{2!} S_\phi''(0) r^2 + \cdots + \frac{1}{(2k)!} S_\phi^{(2k)}(0) r^{2k} + \cdots \\ &= \sum_{k=0}^{\infty} \frac{\Delta^k \phi(0) r^{2k}}{2^k k! m(m+2) \cdots (m+2k-2)} \end{aligned} \quad (2.13)$$

which is the well-known Pizetti's formula.

3. The product r^{-k} and $\nabla\delta$. The following normalization procedure is needed in the proof of our theorem regarding the product of r^{-k} and $\nabla\delta$.

THE DISTRIBUTION $\mu(x)x_+^\lambda$. Let $\mu(x)$ be an infinitely differentiable function on $\mathbb{R}^+$ having properties:

- (i) $\mu(x) \geq 0$,
- (ii) $\mu(0) \neq 0$,
- (iii) $\mu(x) = 0$ for $x \geq 1$.

Let $\phi(x)$ be a testing function. Then the functional

$$(\mu(x)x_+^\lambda, \phi) = \int_0^1 \mu(x)x^\lambda \phi(x) dx \quad (3.1)$$

is regular for $\operatorname{Re} \lambda > -1$. It can be extended to the domain $\operatorname{Re} \lambda > -n-1$ ($\lambda \neq -1, -2, \dots$) by analytic continuation along Gelfand and Shilov (see [5]):

$$\begin{aligned} (\mu(x)x_+^\lambda, \phi) &= \int_0^1 \mu(x)x^\lambda \phi(x) dx \\ &= \sum_{k=1}^n \frac{\phi^{(k-1)}(0) \mu(\theta_{k-1})}{(k-1)! (\lambda+k)} \\ &\quad + \int_0^1 \mu(x)x^\lambda \left[\phi(x) - \phi(0) - x\phi'(0) - \cdots - \frac{x^{n-1}}{(n-1)!} \phi^{(n-1)}(0) \right] dx \end{aligned} \quad (3.2)$$

on applying the mean value theorem with $0 < \theta_{k-1} < 1$ for $1 \leq k \leq n$. This means that the generalized function $\mu(x)x_+^\lambda$ is well defined for $\lambda \neq -1, -2, \dots$

We thus normalize the value of the functional $(\mu(x)x_+^\lambda, \phi)$ at $-n$ by

$$\begin{aligned} (\mu(x)x_+^{-n}, \phi) &= \sum_{k=1}^{n-1} \frac{\phi^{(k-1)}(0)\mu(\theta_{k-1})}{(k-1)!(-n+k)} \\ &\quad + \int_0^1 \mu(x)x^{-n} \left[\phi(x) - \phi(0) - x\phi'(0) - \cdots - \frac{x^{n-1}}{(n-1)!}\phi^{(n-1)}(0) \right] dx. \end{aligned} \quad (3.3)$$

THEOREM 3.1. *The noncommutative neutrix product $r^{-k} \cdot \nabla \delta$ exists. Furthermore*

$$\begin{aligned} r^{-2k} \nabla \delta &= -\frac{1}{2^{k+1}(k+1)!(m+2) \cdots (m+2k)} \sum_{i=1}^m (x_i \triangle^{k+1} \delta), \\ r^{1-2k} \cdot \nabla \delta &= 0, \end{aligned} \quad (3.4)$$

where k is a positive integer and ∇ is the gradient operator.

PROOF. Since $\nabla = \partial/\partial x_1 + \cdots + \partial/\partial x_m = \sum_{i=1}^m \partial/\partial x_i$, we have

$$\nabla \delta_n(x) = 2C_m n^{m+2} \sum_{i=1}^m \rho'(n^2 r^2) x_i = 2C_m n^{m+2} \rho'(n^2 r^2) \sum_{i=1}^m x_i. \quad (3.5)$$

We note that r^{-k} is a locally summable function on $\mathbb{R}^m$ for $k = 1, 2, \dots, m-1$. Therefore

$$\begin{aligned} I &= (r^{-k} \cdot \nabla \delta_n, \phi) = \int_{\mathbb{R}^m} r^{-k} \nabla \delta_n(x) \phi(x) dx \\ &= 2C_m n^{m+2} \sum_{i=1}^m \int_{\mathbb{R}^m} r^{-k} \rho'(n^2 r^2) x_i \phi(x) dx. \end{aligned} \quad (3.6)$$

On changing to spherical polar coordinates and then making the substitution $t = nr$, we arrive at

$$\begin{aligned} I &= 2C_m \Omega_m n^{m+2} \sum_{i=1}^m \int_0^{1/n} r^{m-k-1} \rho'(n^2 r^2) S_{\psi_i}(r) dr \\ &= 2C_m \Omega_m n^{k+2} \sum_{i=1}^m \int_0^1 t^{m-k-1} \rho'(t^2) S_{\psi_i} \left(\frac{t}{n} \right) dt, \end{aligned} \quad (3.7)$$

where $\psi_i(x) = x_i \phi(x)$. By Taylor's formula, we obtain

$$S_{\psi_i}(r) = \sum_{j=0}^{k+1} \frac{S_{\psi_i}^{(j)}(0)}{j!} r^j + \frac{S_{\psi_i}^{(k+2)}(0)}{(k+2)!} r^{k+2} + \frac{S_{\psi_i}^{(k+3)}(\zeta r)}{(k+3)!} r^{k+3}, \quad (3.8)$$

where $0 < \zeta < 1$. Hence

$$\begin{aligned}
 I &= 2C_m \Omega_m n^{m+2} \sum_{i=1}^m \sum_{j=0}^{k+1} \frac{S_{\psi_i}^{(j)}(0)}{j!} \int_0^{1/n} r^{m-k-1} \rho'(n^2 r^2) r^j dr \\
 &\quad + 2C_m \Omega_m n^{m+2} \sum_{i=1}^m \int_0^{1/n} r^{m-k-1} \rho'(n^2 r^2) \frac{S_{\psi_i}^{(k+2)}(0)}{(k+2)!} r^{k+2} dr \\
 &\quad + 2C_m \Omega_m n^{m+2} \sum_{i=1}^m \int_0^{1/n} r^{m-k-1} \rho'(n^2 r^2) \frac{S_{\psi_i}^{(k+3)}(\zeta r)}{(k+3)!} r^{k+3} dr \\
 &= I_1 + I_2 + I_3,
 \end{aligned} \tag{3.9}$$

respectively. Employing $t = nr$ again, we get

$$I_1 = 2C_m \Omega_m \sum_{i=1}^m \sum_{j=0}^{k+1} n^{k+2-j} \frac{S_{\psi_i}^{(j)}(0)}{j!} \int_0^1 t^{m+j-k-1} \rho'(t^2) dt \tag{3.10}$$

whence

$$N - \lim_{n \rightarrow \infty} I_1 = 0 \tag{3.11}$$

as for

$$I_2 = 2C_m \Omega_m \sum_{i=1}^m \frac{S_{\psi_i}^{(k+2)}(0)}{(k+2)!} \int_0^1 t^{m+1} \rho'(t^2) dt \tag{3.12}$$

integrating by parts, we have

$$\begin{aligned}
 2C_m \Omega_m \int_0^1 t^{m+1} \rho'(t^2) dt &= C_m \Omega_m \int_0^1 t^m d\rho(t^2) \\
 &= -C_m \Omega_m \cdot m \int_0^1 t^{m-1} \rho(t^2) dt \\
 &= -m \int_{\mathbb{R}^m} \delta_n(x) dx = -m.
 \end{aligned} \tag{3.13}$$

Hence

$$I_2 = -m \sum_{i=1}^m \frac{S_{\psi_i}^{(k+2)}(0)}{(k+2)!} = -\frac{m}{(k+2)!} \sum_{i=1}^m S_{\psi_i}^{(k+2)}(0). \tag{3.14}$$

Putting

$$M = \sup \left\{ \left| S_{\psi_i}^{(k+3)}(r) \right| : r \in \mathbb{R}^+ \text{ and } 1 \leq i \leq m \right\}, \tag{3.15}$$

we obtain that

$$|I_3| \leq 2C_m \Omega_m \frac{mM}{n(k+3)!} \int_0^1 t^{m+2} |\rho'(t^2)| dt \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{3.16}$$

Hence it follows from above that

$$N - \lim_{n \rightarrow \infty} I = I_2 = -\frac{m}{(k+2)!} \sum_{i=1}^m S_{\psi_i}^{(k+2)}(0). \tag{3.17}$$

We now turn our attention to the product $r^{-k} \cdot \nabla \delta$ for $k \geq m$. Note that, in this case, the functional r^{-k} is not locally summable. We assume $k = m + q$ for $q = 0, 1, 2, \dots$, then $-k + m - 1 \leq -1$. We apply the regularization in (3.3) to I of (3.7) to deduce

$$\begin{aligned}
 I &= 2C_m \Omega_m n^{k+2} \sum_{i=1}^m \left\{ \sum_{j=1}^{q=k-m} \frac{S_{\psi_i}^{(j-1)}(0) \rho'(\theta_{j-1}^2)}{(j-1)!(m-k-1+j)} \quad (= I_1) \right. \\
 &\quad \left. + \int_0^1 \rho'(t^2) t^{m-k-1} \right. \\
 &\quad \left. \times \left[S_{\psi_i} \left(\frac{t}{n} \right) - S_{\psi_i}(0) - \dots - \frac{t^q}{n^q q!} S_{\psi_i}^{(q)}(0) \right] dt \right\} \quad (= I_2) \\
 &= I_1 + I_2,
 \end{aligned} \tag{3.18}$$

respectively.

Clearly,

$$N - \lim_{n \rightarrow \infty} I_1 = 0. \tag{3.19}$$

Applying Taylor's theorem, we obtain

$$\begin{aligned}
 I_2 &= 2C_m \Omega_m n^{k+2} \sum_{i=1}^m \int_0^1 \rho'(t^2) t^{m-k-1} \left[\frac{t^{q+1}}{n^{q+1}(q+1)!} S_{\psi_i}^{(q+1)}(0) + \dots \right. \\
 &\quad \left. + \frac{t^{q+m+2}}{n^{q+m+2}(q+m+2)!} S_{\psi_i}^{(q+m+2)}(0) \right. \\
 &\quad \left. + \frac{t^{q+m+3}}{n^{q+m+3}(q+m+3)!} S_{\psi_i}^{(q+m+3)} \left(\frac{\theta t}{n} \right) \right] dt,
 \end{aligned} \tag{3.20}$$

where $0 < \theta < 1$. Similarly, we could prove

$$\begin{aligned}
 N - \lim_{n \rightarrow \infty} I_2 &= 2C_m \Omega_m \int_0^1 \rho'(t^2) t^{m+1} dt \sum_{i=1}^m \frac{S_{\psi_i}^{(q+m+2)}(0)}{(q+m+2)!} \\
 &= -\frac{m}{(q+m+2)!} \sum_{i=1}^m S_{\psi_i}^{(q+m+2)}(0) \\
 &= -\frac{m}{(k+2)!} \sum_{i=1}^m S_{\psi_i}^{(k+2)}(0)
 \end{aligned} \tag{3.21}$$

because the other terms vanish upon taking their N -limits.

Using Pizetti's formula, we get

$$S_{\psi_i}^{(k+2)}(0) = \begin{cases} \frac{(2l+2)! \triangle^{l+1} \psi_i(0)}{2^{l+1} (l+1)! m(m+2) \cdots (m+2l)} & \text{if } k = 2l \text{ for } l = 0, 1, \dots, \\ 0 & \text{if } k = 2l - 1 \text{ for } l = 1, 2, \dots \end{cases} \tag{3.22}$$

This completes the proof. $\square$

REMARK 3.2. The multiplication of x_i and $\Delta^{k+1}\delta$ in our theorem is well defined since

$$(x_i \Delta^{k+1}\delta, \phi) = (\delta, \Delta^{k+1}(x_i \phi)). \quad (3.23)$$

In particular, we have the following

$$\frac{1}{x^2} \cdot \delta'(x) = \frac{1}{6} \delta^{(3)}(x) \quad (3.24)$$

by setting $m = 1$ and $k = 1$ in the theorem, which identically coincides with equation (1.7) with $m = 1$, $r = 2$, and $p = 1$.

ACKNOWLEDGEMENT. The author is grateful to Professor D. Grant who made several productive suggestions, which improved the quality of this paper. This research is supported by a UCCB research grant.

REFERENCES

- [1] L. Z. Cheng and C. K. Li, *A commutative neutrix product of distributions on $\mathbb{R}^m$* , Math. Nachr. **151** (1991), 345–355. MR 92j:46067. Zbl 743.46026.
- [2] B. Fisher, *On defining the convolution of distributions*, Math. Nachr. **106** (1982), 261–269. MR 83m:46066. Zbl 499.46024.
- [3] B. Fisher and C. K. Li, *On defining a noncommutative product of distributions in m variables*, J. Natur. Sci. Math. **31** (1991), no. 2, 95–102. MR 93a:46073. Zbl 772.46016.
- [4] S. Gasiorowicz, *Elementary Particle Physics*, vol. 20, John Wiley & Sons, New York, London, Sydney, 1967. Zbl 166.23505.
- [5] I. M. Gel'fand and G. E. Shilov, *Generalized Functions. Vol. 1*, Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1964 [1977]. MR 55#8786a. Zbl 115.33101.
- [6] J. G. van der Corput, *Introduction to the neutrix calculus*, J. Analyse Math. **7** (1959/1960), 281–399. MR 23#A1989. Zbl 097.10503.

C. K. LI: UNIVERSITY COLLEGE OF CAPE BRETON, MATHEMATICS AND COMPUTER SCIENCE,
 SYDNEY, NOVA SCOTIA, CANADA B1P 6L2
E-mail address: cli@uccb.ns.ca