A SUBORDINATION THEOREM FOR SPIRALLIKE FUNCTIONS

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ABSTRACT. We prove a subordination relation for a subclass of the class of λ -spirallike functions.

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1. Introduction. Let K denote the usual class of convex functions. Denote by $S_p(\lambda)$, $-\pi/2 < \lambda < \pi/2$, the class of functions $f(z) = z + a_2 z^2 + \cdots$ which are analytic in E and satisfy therein the condition

$$\operatorname{Re} \left[e^{i\lambda} \frac{zf'(z)}{f(z)} \right] > 0. \quad (1.1)$$

Spacek [3] proved that members of $S_p(\lambda)$, known as λ spirallike functions, are univalent in E . In 1989, Silverman [2] proved that if

$$\sum_{n=2}^{\infty} [1 + (n-1) \sec \lambda] |a_n| \leq 1 \quad \left(|\lambda| < \frac{\pi}{2} \right), \quad (1.2)$$

then the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belongs to $S_p(\lambda)$. Let us denote by $G(\lambda)$, the class of function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ whose coefficients satisfy the condition (1.2). Note that $G(0)$ is a subclass of the class of starlike functions (with respect to the origin) (see Silverman [1]).

In this paper, we prove a subordination theorem for the class $G(\lambda)$. To state and prove our main result we need the following definitions and lemma.

DEFINITION 1.1. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ are analytic in $|z| < r$, then their Hadamard product/convolution, $f * g$ is the function defined by the power series

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n. \quad (1.3)$$

The function $f * g$ is also analytic in $|z| < r$.

DEFINITION 1.2. Let f be analytic in E , g analytic and univalent in E and $f(0) = g(0)$. Then by the symbol $f(z) \prec g(z)$ (f subordinate to g) in E , we shall mean that $f(E) \subset g(E)$.

DEFINITION 1.3. A sequence $\{b_n\}_1^\infty$ of complex numbers is said to be a subordinating factor sequence if whenever $f(z) = \sum_{k=1}^\infty a_k z^k$, $a_1 = 1$ is regular, univalent and convex in E , we have

$$\sum_{k=1}^\infty b_k a_k z^k < f(z) \quad \text{in } E. \quad (1.4)$$

LEMMA 1.4. The sequence $\{b_n\}_1^\infty$ is a subordinating factor sequence if and only if

$$\operatorname{Re} \left[1 + 2 \sum_{n=1}^\infty b_n z^n \right] > 0, \quad (z \in E). \quad (1.5)$$

This lemma which gives a beautiful characterisation of a subordinating factor sequence is due to Wilf [4].

2. Main theorem

THEOREM 2.1. Let $f \in G(\lambda)$. Then

$$\frac{1 + \sec \lambda}{2(2 + \sec \lambda)} (f * g)(z) < g(z), \quad (z \in E) \quad (2.1)$$

for every function g in the class K .

In particular

$$\operatorname{Re} f(z) > -\frac{2 + \sec \lambda}{(1 + \sec \lambda)}, \quad (z \in E). \quad (2.2)$$

The constant $(1 + \sec \lambda)/2(2 + \sec \lambda)$ cannot be replaced by any larger one.

Taking $\lambda = 0$, we obtain the following corollary.

COROLLARY 2.2. If $f(z) = z + a_2 z^2 + \dots$ is regular in E and satisfies therein the condition

$$\sum_{n=2}^\infty n |a_n| \leq 1, \quad (2.3)$$

then for every function g in K , we have

$$\frac{1}{3} (f * g)(z) < g(z), \quad (|z| < 1). \quad (2.4)$$

In particular, $\operatorname{Re} f(z) > -3/2$, $z \in E$. The constant $1/3$ is best possible.

PROOF OF THEOREM 2.1. Let $f(z) = z + \sum_{n=2}^\infty a_n z^n$ be any member of the class $G(\lambda)$ and let $g(z) = z + \sum_{n=2}^\infty c_n z^n$ be any function in the class K . Then

$$\frac{1 + \sec \lambda}{2(2 + \sec \lambda)} (f * g)(z) = \frac{1 + \sec \lambda}{2(2 + \sec \lambda)} \left(z + \sum_{n=2}^\infty a_n c_n z^n \right). \quad (2.5)$$

Thus, by Definition 1.3, the assertion of our theorem will hold if the sequence

$$\left(\frac{(1 + \sec \lambda) a_n}{2(2 + \sec \lambda)} \right)_{n=1}^\infty \quad (2.6)$$

is a subordinating factor sequence, with $a_1 = 1$. In view of the lemma, this will be the

case if and only if

$$\operatorname{Re} \left[1 + 2 \sum_{n=1}^{\infty} \frac{1 + \sec \lambda}{2(2 + \sec \lambda)} a_n z^n \right] > 0, \quad (z \in E). \quad (2.7)$$

Now

$$\begin{aligned} & \operatorname{Re} \left[1 + \frac{1 + \sec \lambda}{2 + \sec \lambda} \sum_{n=1}^{\infty} a_n z^n \right] \\ &= \operatorname{Re} \left[1 + \frac{1 + \sec \lambda}{2 + \sec \lambda} z + \frac{1}{2 + \sec \lambda} \sum_{n=2}^{\infty} (1 + \sec \lambda) a_n z^n \right] \\ &> \left[1 - \frac{1 + \sec \lambda}{2 + \sec \lambda} r - \frac{1}{2 + \sec \lambda} \sum_{n=2}^{\infty} (1 + (n-1) \sec \lambda) |a_n| r^n \right] \quad (2.8) \\ &\quad (\text{because } 1 + \sec \lambda \leq 1 + (n-1) \sec \lambda \text{ for all } n \geq 2, |\lambda| < \pi/2) \\ &> \left[1 - \frac{1 + \sec \lambda}{2 + \sec \lambda} r - \frac{1}{2 + \sec \lambda} r \right] \quad (|z| = r) \\ &> 0. \end{aligned}$$

Thus (2.7) holds true in E . This proves the first assertion. That $\operatorname{Re} f(z) > -(2 + \sec \lambda)/(1 + \sec \lambda)$ for $f \in G(\lambda)$ follows by taking $g(z) = z/(1 - z)$ in (2.1). To prove the sharpness of the constant $(1 + \sec \lambda)/2(2 + \sec \lambda)$, we consider the function f_0 defined by $f_0(z) = z - (1/(1 + \sec \lambda))z^2$ ($|\lambda| < \pi/2$), which is a member of the class $G(\lambda)$. Thus from the relation (2.1) we obtain

$$\frac{1 + \sec \lambda}{2(2 + \sec \lambda)} f_0(z) < \frac{z}{1 - z}. \quad (2.9)$$

It can be easily verified that

$$\min_{|z| \leq 1} \operatorname{Re} \left[\frac{1 + \sec \lambda}{2(2 + \sec \lambda)} f_0(z) \right] = -\frac{1}{2}. \quad (2.10)$$

This shows that the constant $(1 + \sec \lambda)/2(2 + \sec \lambda)$ is best possible. $\square$

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