FREE MINIMAL RESOLUTIONS AND THE BETTI NUMBERS OF THE SUSPENSION OF AN n -GON

TILAK DE ALWIS

(Received 31 July 1998)

ABSTRACT. Consider the general n -gon with vertices at the points $1, 2, \dots, n$. Then its suspension involves two more vertices, say at $n+1$ and $n+2$. Let R be the polynomial ring $k[x_1, x_2, \dots, x_n]$, where k is any field. Then we can associate an ideal I to our suspension in the Stanley-Reisner sense. In this paper, we find a free minimal resolution and the Betti numbers of the R -module R/I .

Keywords and phrases. Suspension, finite abstract simplicial complex, Stanley-Reisner ideal, free-minimal resolution, Betti numbers, double complex.

2000 Mathematics Subject Classification. Primary 13H10.

1. Introduction. Consider the suspension of the n -gon whose vertices are at the points $1, 2, \dots, n$ (see [6]). This introduces two new vertices, say $n+1$ and $n+2$. The finite abstract simplicial complex Ω corresponding to this suspension is given by

$$\begin{aligned} \Omega = \{ \emptyset, \{1\}, \{2\}, \dots, \{n\}, \{n+1\}, \{n+2\}, \{1, 2\}, \{2, 3\}, \dots, \{n, 1\}, \\ \{1, n+1\}, \{2, n+1\}, \dots, \{n-1, n+1\}, \{n, n+1\}, \{1, n+2\}, \\ \{2, n+2\}, \dots, \{n-1, n+2\}, \{n, n+2\} \}. \end{aligned} \quad (1.1)$$

Let k be any field and $R = k[x_1, \dots, x_{n+2}]$. By definition, the Stanley-Reisner ideal associated to Ω is given by $I =$ The ideal in R generated by all the monomials of the form $x_{i_1} x_{i_2} \cdots x_{i_r}$, where $1 \leq i_1 < i_2 < \cdots < i_r \leq n+2$ and $\{i_1, \dots, i_r\} \notin \Omega$ (see [3, 7]). Then, it follows that $I = (x_1 x_3, x_1 x_4, \dots, x_1 x_{n-1}, x_2 x_4, \dots, x_2 x_n, \dots, x_{n-2} x_n, x_{n+1} x_{n+2})$ for $n > 3$, and $I = (x_1 x_2 x_3, x_4 x_5)$ for $n = 3$. In the literature, the ring R/I is also known as the face ring or the Stanley-Reisner ring of the finite abstract simplicial complex Ω (see [3, 7]).

By definition, a free-minimal resolution of the R -module R/I is an exact sequence of the form

$$\cdots M_i \rightarrow \cdots \rightarrow M_1 \rightarrow M_0 \rightarrow \frac{R}{I} \rightarrow 0, \quad (1.2)$$

where each M_i is a free R -module with the smallest possible rank. For material on free-minimal resolutions, the reader can refer to [5] or [7]. The Betti numbers $B_i(n)$ of the R -module R/I are just the ranks of those free modules M_i , i.e., $B_i(n) = \text{rank}_R(M_i)$ for $i = 0, 1, \dots$

In this paper, we find a free-minimal resolution and the Betti numbers of the R -module R/I . Sometimes we simply refer to them as a free-minimal resolution and the Betti numbers of the suspension of the n -gon.

2. Some useful results. In this section, we recall some results on free-minimal resolutions and the Betti numbers of the n -gon. These results are needed to obtain the theorems on the suspension of the n -gon. The proofs of most of these theorems can be found in [1] or [2].

(1) Let Δ be the finite abstract simplicial complex corresponding to the n -gon with vertices at the points $1, 2, \dots, n$. Let $S = k[x_1, \dots, x_n]$ and J_1 be the Stanley-Reisner ideal associated to Δ . Then, it easily follows that $J_1 = (x_1x_3, x_1x_4, \dots, x_2x_4, \dots, x_2x_n, \dots, x_{n-2}x_n)$ for $n > 3$, and $J_1 = (x_1x_2x_3)$ for $n = 3$.

(2) Let $\beta_i(n)$ denote the i th Betti number of the S -module S/J_1 . In other words, it is the i th Betti number of the n -gon. Then, for $n \geq 3$,

$$\beta_i(n) = \begin{cases} 1, & i = 0, \\ \binom{n}{i+1} \frac{i(n-i-2)}{n-1}, & i = 1, 2, \dots, n-3, \\ 1, & i = n-2, \\ 0, & \text{otherwise.} \end{cases} \quad (2.1)$$

(3) We can show that,

$$0 \longrightarrow S^{\beta_{n-2}} \xrightarrow{f_{n-2}} S^{\beta_{n-3}} \longrightarrow \dots \longrightarrow S^{\beta_1} \xrightarrow{f_1} S^{\beta_0} \xrightarrow{f_0} \frac{S}{J_1} \longrightarrow 0 \quad (2.2)$$

is a free-minimal resolution of the S -module S/J_1 . Even though we do not need the specific definitions of the maps f_j for what follows, the inquisitive reader can find them in [1].

3. Main results. Let J_1 be the ideal in the polynomial ring $S = k[x_1, \dots, x_n]$ as in Section 2. Let J be the ideal in the polynomial ring $R = k[x_1, \dots, x_n, x_{n+1}, x_{n+2}]$ generated by the same generators as that of J_1 .

Tensor the exact sequence (2.2) with the k -module $k[x_{n+1}, x_{n+2}]$, which is a free module. Hence we obtain the following exact sequence of R -modules.

$$0 \longrightarrow R^{\beta_{n-2}} \xrightarrow{d_{n-2}} R^{\beta_{n-3}} \longrightarrow \dots \longrightarrow R^{\beta_1} \xrightarrow{d_1} R^{\beta_0} \xrightarrow{d_0} \frac{R}{J} \longrightarrow 0, \quad (3.1)$$

where d_i are the same as the maps $f_i \otimes id$. This means that the following complex is exact at all places except at degree 0:

$$\begin{array}{ccccccc} n-2 & & n-3 & & 1 & & 0 \\ 0 & \longrightarrow & R^{\beta_{n-2}} & \xrightarrow{d_{n-2}} & R^{\beta_{n-3}} & \longrightarrow & \dots \longrightarrow R^{\beta_1} \xrightarrow{d_1} R^{\beta_0} \longrightarrow 0. \end{array} \quad (3.2)$$

Consider the following diagram where the two rows are the same as the complex (3.2) and the vertical maps are multiplication by the element $y = x_{n+1}x_{n+2}$ of R .

$$\begin{array}{ccccccc}
 0 & \longrightarrow & R^{\beta_{n-2}} & \xrightarrow{d_{n-2}} & R^{\beta_{n-3}} & \longrightarrow \cdots \longrightarrow & R^{\beta_1} \xrightarrow{d_1} R^{\beta_0} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow & \downarrow \\
 0 & \longrightarrow & R^{\beta_{n-2}} & \xrightarrow{d_{n-2}} & R^{\beta_{n-3}} & \longrightarrow \cdots \longrightarrow & R^{\beta_1} \xrightarrow{d_1} R^{\beta_0} \longrightarrow 0.
 \end{array} \quad (3.3)$$

The squares in (3.3) commute, because $x_{n+1}x_{n+2}$ is an element of the ring R and our maps are R -module homomorphisms. Hence (3.3) is a double complex, and its total complex is given by

$$0 \longrightarrow R^{\beta_{n-2}} \xrightarrow{\partial_{n-1}} R^{\beta_{n-2}} \oplus R^{\beta_{n-3}} \longrightarrow \cdots \longrightarrow R^{\beta_1} \oplus R^{\beta_0} \xrightarrow{\partial_1} R^{\beta_0} \longrightarrow 0, \quad (3.4)$$

where the differential maps $\partial_i : R^{\beta_i} \oplus R^{\beta_{i-1}} \rightarrow R^{\beta_{i-1}} \oplus R^{\beta_{i-2}}$, $i = 1, 2, \dots, n-1$ are given by $\partial_i(p, q) = (d_i(p) + (-1)^i yq, d_{i-1}(q))$ for $i = 2, 3, \dots, n-2$. Obvious definitions would apply for $i = 1$ and $i = n-1$. It is a routine exercise to verify that $\partial_{i-1} \circ \partial_i = 0$.

THEOREM 3.1. *The complex (3.4) is exact at all places except degree 0 at which it has homology equal to R/I . In other words, the following is a free resolution of R/I :*

$$0 \longrightarrow R^{\beta_{n-2}} \xrightarrow{\partial_{n-1}} R^{\beta_{n-2}} \oplus R^{\beta_{n-3}} \longrightarrow \cdots \longrightarrow R^{\beta_1} \oplus R^{\beta_0} \xrightarrow{\partial_1} R^{\beta_0} \longrightarrow \frac{R}{I} \longrightarrow 0. \quad (3.5)$$

PROOF. Denote R^{β_i} by D_i . Then for $i > 1$, consider the sequence $D_{i+1} \oplus D_i \xrightarrow{\partial_{i+1}} D_i \oplus D_{i-1} \xrightarrow{\partial_i} D_{i-1} \oplus D_{i-2}$. Suppose $(p, q) \in \text{Ker } \partial_i$. Then $\partial_i(p, q) = (d_i(p) + (-1)^i yq, d_{i-1}(q)) = 0$. Hence $d_i(p) + (-1)^i yq = 0$ and $d_{i-1}(q) = 0$. Therefore, $q \in \text{Ker } d_{i-1} = \text{im } d_i$, so $q = d_i(q_1)$ for some $q_1 \in D_i$. The equation $d_i(p) + (-1)^i yq = 0$ yields $d_i(p + (-1)^i yq_1) = 0$, which means that $p + (-1)^i yq_1 \in \text{Ker } d_i = \text{im } d_{i+1}$. Therefore, $p + (-1)^i yq_1 = d_{i+1}(p_1)$ for some $p_1 \in D_{i+1}$, which implies that $p = d_{i+1}(p_1) + (-1)^{i+1} yq_1$. Hence, $\partial_{i+1}(p_1, q_1) = (p, q)$, i.e., $(p, q) \in \text{im } \partial_{i+1}$. This shows that $\text{Ker } \partial_i = \text{im } \partial_{i+1}$ for $i > 1$.

For $i = 1$, we have $D_2 \oplus D_1 \xrightarrow{\partial_2} D_1 \oplus D_0 \xrightarrow{\partial_1} D_0$. Let $(p, q) \in \text{Ker } \partial_1$. Therefore, $\partial_1(p, q) = d_1(p) + (-1)yq = 0$. This yields $d_1(p) = yq \in \text{im } d_1 = \text{Ker } d_0 = J$. But $y \notin J$. Hence, even though J is not a prime ideal of R , by considering the primary decomposition of J , one can easily obtain that $q \in J$. Therefore, the exact sequence (3.1) gives us $q = d_1(q'_1)$ for some $q'_1 \in D_1$. Hence $d_1(p) = yq = yd_1(q'_1) = d_1(yq'_1)$, which implies that $p - yq'_1 \in \text{Ker } d_1 = \text{im } d_2$. Therefore, $p - yq'_1 = d_2(p'_1)$ for some $p'_1 \in D_2$. Hence, $p = d_2(p'_1) + yq'_1$. Now we have two equations $d_2(p'_1) + yq'_1 = p$, and $d_1(q'_1) = q$ where $(p'_1, q'_1) \in D_2 \oplus D_1$. This yields $\partial_2(p'_1, q'_1) = (p, q)$ and hence $\text{Ker } \partial_1 = \text{im } \partial_2$.

Finally, for $i = 0$, we have $D_1 \oplus D_0 \xrightarrow{\partial_1} D_0 \rightarrow 0$. We know that $\partial_1(p, q) = d_1(p) - yq$. However, the exact sequence (3.1) implies that $d_1(D_1) = J$ and hence $\text{im } \partial_1 = \{j - yq \mid j \in J, q \in R\} = J + (y) = I$. Therefore the homology of the complex (3.4) at the zeroth spot is equal to R/I . $\square$

Theorem 3.2 says more about the free resolution (3.5).

THEOREM 3.2. *The sequence (3.5) is a free-minimal resolution of the R -module R/I .*

PROOF. To show the minimality, it is enough to show that the maps $\partial_i \otimes id : (R^{\beta_i} \oplus R^{\beta_{i-1}}) \otimes_R k \rightarrow (R^{\beta_{i-1}} \oplus R^{\beta_{i-2}}) \otimes_R k$ are zero for $i = 1, 2, \dots, n-1$ (see [4, p. 136]). However, this is an easy consequence of commutativity of the following diagram and the minimality of (2.2):

$$\begin{array}{ccc} (R^{\beta_i} \oplus R^{\beta_{i-1}}) \otimes_R k & \longrightarrow & (R^{\beta_{i-1}} \oplus R^{\beta_{i-2}}) \otimes_R k \\ \downarrow & & \downarrow \\ (R^{\beta_i} \otimes_R k) \oplus (R^{\beta_{i-1}} \otimes_R k) & \longrightarrow & (R^{\beta_{i-1}} \otimes_R k) \oplus (R^{\beta_{i-2}} \otimes_R k). \end{array} \quad (3.6) \quad \square$$

Theorem 3.3 enables us to calculate the Betti numbers $B_i(n)$ of the suspension of the n -gon.

THEOREM 3.3. *Let $n \geq 3$ be a positive integer. Then the i th Betti number $B_i(n)$ of the suspension of the n -gon is given by*

$$B_i(n) = \begin{cases} 1, & i = 0, \\ \binom{n-1}{2}, & i = 1, \\ \binom{n}{i} \frac{[ni - (i^2 + i + 1)]}{i+1}, & i = 2, 3, \dots, n-3, \\ \binom{n-1}{2}, & i = n-2, \\ 1, & i = n-1, \\ 0, & \text{otherwise.} \end{cases} \quad (3.7)$$

PROOF. Let $n \geq 3$ be a positive integer. Since (3.5) is a free-minimal resolution, the Betti numbers of R/I are just the respective ranks of the free modules appearing in (3.5). Hence, we obtain, for $n \geq 3$,

$$B_i(n) = \begin{cases} \beta_0, & i = 0, \\ \beta_i + \beta_{i-1}, & i = 1, 2, \dots, n-2, \\ \beta_{n-2}, & i = n-1. \end{cases} \quad (3.8)$$

Let us denote $B_i(n)$ by B_i . The theorem is clear for $n = 3$. Therefore assume that $n > 3$. So

$$B_0 = \beta_0 = \beta_{n-2} = B_{n-1} = 1 \quad (3.9)$$

and

$$B_1 = \beta_1 + \beta_0 = \binom{n}{2} \frac{n-3}{n-1} + 1 = \frac{1}{2}n(n-3) + 1 = \frac{1}{2}(n-1)(n-2) = \binom{n-1}{2} \quad (3.10)$$

by using formula (2.1). A similar calculation shows that $B_{n-2} = \binom{n-1}{2}$. Now, let $1 < i < n-2$. The formula (2.1) again gives,

$$\begin{aligned}
 B_i &= \beta_i + \beta_{i-1} = \binom{n}{i+1} \frac{i(n-i-2)}{n-1} + \binom{n}{i} \frac{(i-1)(n-i+1-2)}{n-1} \\
 &= \frac{n!}{(i+1)!(n-i-1)!} \cdot \frac{i(n-i-2)}{n-1} + \frac{n!}{i!(n-i)!} \cdot \frac{(i-1)(n-i-1)}{n-1} \\
 &= \frac{n!}{(n-1)(i+1)(n-i)i!(n-i-1)!} \\
 &\quad \times [i(n-i)(n-i-2) + (i-1)(i+1)(n-i-1)] \quad (3.11) \\
 &= \frac{n!}{(n-1)(i+1)(n-i)i!(n-i-1)!} [n^2i - n(i+1)^2 + i^2 + i + 1] \\
 &= \frac{n!}{(n-1)(i+1)(n-i)i!(n-i-1)!} (n-1) [ni - (i^2 + i + 1)] \\
 &= \frac{n!}{i!(n-i)!} \frac{[ni - (i^2 + i + 1)]}{i+1} = \binom{n}{i} \frac{[ni - (i^2 + i + 1)]}{i+1}
 \end{aligned}$$

which proves the formula (3.7) for the case $n > 3$. This completes the proof. $\square$

We can illustrate our theory with an example. For $n = 4$ we get the suspension of the square, which is nothing but the familiar octahedron. Hence,

$$\begin{aligned}
 \Omega &= \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}, \\
 &\quad \{1, 5\}, \{2, 5\}, \{3, 5\}, \{4, 5\}, \{1, 6\}, \{2, 6\}, \{3, 6\}, \{4, 6\}\}, \\
 R &= k[x_1, x_2, x_3, x_4, x_5, x_6], \\
 I &= (x_1x_3, x_2x_4, x_5x_6). \quad (3.12)
 \end{aligned}$$

The formula (3.7) gives us $B_0(4) = 1$, $B_1(4) = 3$, $B_2(4) = 3$, and $B_3(4) = 1$ which are the Betti numbers of the octahedron.

ACKNOWLEDGEMENT. The author wishes to thank his adviser John Eagon for all the valuable help.

REFERENCES

- [1] T. de Alwis, *Free minimal resolutions*, Comm. Algebra **21** (1993), no. 12, 4575-4585. MR 95f:13017. Zbl 806.18009.
- [2] ———, *Pascal type properties of Betti numbers*, Internat. J. Math. Math. Sci. **17** (1994), no. 3, 545-552. MR 95c:13026. Zbl 809.13010.
- [3] M. Hochster, *Cohen-Macaulay rings, combinatorics, and simplicial complexes*, Ring theory, II (New York), Lecture Notes in Pure and Appl. Math., vol. 26, Proc. Second Conf., Univ. Oklahoma, Dekker, 1977, pp. 171-223. MR 56 376. Zbl 351.13009.
- [4] H. Matsumura, *Commutative algebra*, 2d ed., Mathematics Lecture Note Series, vol. 56, Benjamin/Cummings Publishing Co., Inc., Reading, M.S., 1980. MR 82i:13003. Zbl 441.13001.
- [5] ———, *Commutative ring theory*, Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, New York, 1986. MR 88h:13001. Zbl 603.13001.

- [6] J. R. Munkres, *Elements of Algebraic Topology*, Addison-Wesley Publishing Company, Menlo Park, Calif., 1984. MR 85m:55001. Zbl 673.55001.
- [7] R. P. Stanley, *Combinatorics and commutative algebra*, Progress in Mathematics, vol. 41, Birkhauser Boston, Inc., Boston, M.A., 1983. MR 85b:05002. Zbl 537.13009.

DE ALWIS: DEPARTMENT OF MATHEMATICS, SOUTHEASTERN LOUISIANA UNIVERSITY, HAMMOND, LOUISIANA 70402, USA