ON A CLASS OF UNIVALENT FUNCTIONS

VIKRAMADITYA SINGH

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ABSTRACT. We consider the class of univalent functions $f(z) = z + a_3z^3 + a_4z^4 + \cdots$ analytic in the unit disc and satisfying $|(z^2 f'(z)/f^2(z)) - 1| < 1$, and show that such functions are starlike if they satisfy $|(z^2 f'(z)/f^2(z)) - 1| < (1/\sqrt{2})$.

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Let A denote the class of functions which are analytic in the unit disc $U = \{z : |z| < 1\}$ and have Taylor series expansion

$$f(z) = z + a_2z^2 + a_3z^3 + \cdots, \quad (1)$$

and let T be the univalent [3] subclass of A which satisfy

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < 1, \quad z \in U. \quad (2)$$

By T_2 we denote the subclass of T for which $f''(0) = 0$. In this paper, we prove the following theorem.

THEOREM 1. *If $f \in T_2$, then*

- (i) $\operatorname{Re}(f(z)/z) > 1/2$, $z \in U$,
- (ii) f is starlike in $|z| < 1/\sqrt[4]{2} = 0.840896\dots$,
- (iii) $\operatorname{Re} f'(z) > 0$ for $|z| < 1/\sqrt{2}$.

Items (i) and (iii) are improvements of results in [2], and (ii) is the same as in [2] but has a different proof. Furthermore, (i) and (iii) are sharp as shown by the function

$$f(z) = \frac{z}{1 - z^2}, \quad (3)$$

but the sharpness of (ii) is difficult to establish by a direct example. We also prove the following theorem which partially answers a question raised in [1].

THEOREM 2. *If $T_{2,\mu}$ is the subclass of T_2 which satisfies*

$$\left| z^2 \frac{f'(z)}{f^2(z)} - 1 \right| < \mu < 1, \quad (4)$$

then $T_{2,\mu}$ is a subclass of starlike functions if $0 \leq \mu \leq 1/\sqrt{2}$.

We define by B the class of functions ω analytic in U and satisfying

$$|\omega(z)| < 1, \quad z \in U, \quad \omega(0) = \omega'(0) = 0. \quad (5)$$

From Schwarz's lemma it then follows that

$$|\omega(z)| \leq |z|^2. \quad (6)$$

PROOF OF THEOREM 1. If $f \in T_2$ and satisfies (2), then

$$z^2 \frac{f'(z)}{f^2(z)} - 1 = \omega(z), \quad z \in U, \quad \omega \in B, \quad (7)$$

and by direct integration

$$\frac{z}{f(z)} = 1 - \int_0^1 \frac{\omega(tz)}{t^2} dt, \quad z \in U, \quad \omega \in B. \quad (8)$$

From (8), we obtain

$$\left| \frac{z}{f(z)} - 1 \right| \leq |z|^2 < 1, \quad (9)$$

and this gives

$$\left| 1 - \frac{f(z)}{z} \right| \leq \left| \frac{f(z)}{z} \right|, \quad (10)$$

which is equivalent to $(\operatorname{Re} f(z)/z) > 1/2$, This proves (i).

Furthermore, from (9), we obtain

$$\left| \arg \frac{f(z)}{z} \right| \leq \sin^{-1} |z|^2. \quad (11)$$

From (7), we obtain

$$z \frac{f'(z)}{f(z)} = \frac{f(z)}{z} (1 + \omega(z)) \quad (12)$$

and, therefore,

$$\left| \arg \frac{zf'(z)}{f(z)} \right| = \left| \arg \frac{f(z)}{z} + \arg (1 + \omega(z)) \right| \leq 2 \sin^{-1} |z|^2. \quad (13)$$

This gives (ii).

In order to prove (iii), we notice that (7) yields

$$f'(z) = \left(\frac{f(z)}{z} \right)^2 (1 + \omega(z)) \quad (14)$$

and, therefore,

$$|\arg f'(z)| = \left| 2 \arg \frac{f(z)}{z} + \arg (1 + \omega(z)) \right| \leq 3 \sin^{-1} |z|^2. \quad (15)$$

But this is equivalent to (iii). $\square$

PROOF OF THEOREM 2. If $f \in T_{2,\mu}$, we obtain from (4)

$$z \frac{f'(z)}{f^2(z)} - 1 = \mu \omega(z), \quad \omega \in B, \quad z \in U \quad \text{and} \quad \frac{z}{f(z)} = 1 - \mu \int_0^1 \frac{\omega(tz)}{t^2} dt. \quad (16)$$

Hence

$$z \frac{f'(z)}{f(z)} = \frac{1 + \mu \omega(z)}{1 - \mu \int_0^1 (\omega(tz)/t^2) dt}. \quad (17)$$

Now $\operatorname{Re} z(f'(z)/f(z)) > 0$ is equivalent to the condition

$$z \frac{f'(z)}{f(z)} = \frac{1 + \mu \omega(z)}{1 - \mu \int_0^1 (\omega(tz)/t^2) dt} \neq -iT, \quad T \in \mathbb{R}. \quad (18)$$

Relation (18) is equivalent to

$$\frac{\mu}{2} \left[\left(\omega(z) + \int_0^1 \frac{\omega(tz)}{t^2} dt \right) + \frac{1-iT}{1+iT} \left(\omega(z) - \int_0^1 \frac{\omega(tz)}{t^2} dt \right) \right] \neq -1. \quad (19)$$

Let

$$M = \sup_{z \in U, \omega \in B, T \in \mathbb{R}} \left| \left[\left(\omega(z) + \int_0^1 \frac{\omega(tz)}{t^2} dt \right) + \frac{1-iT}{1+iT} \left(\omega(z) - \int_0^1 \frac{\omega(tz)}{t^2} dt \right) \right] \right|, \quad (20)$$

then, in view of the rotation invariance of B , it follows that

$$\operatorname{Re} z \frac{f'(z)}{f(z)} > 0, \quad \text{if } \mu \leq \frac{2}{M}. \quad (21)$$

However, from (20), we notice that

$$\begin{aligned} M &\leq \sup_{z \in U, \omega \in B} \left[\left| \omega(z) + \int_0^1 \frac{\omega(tz)}{t^2} dt \right| + \left| \omega(z) - \int_0^1 \frac{\omega(tz)}{t^2} dt \right| \right] \\ &\leq 2 \sup_{z \in U, \omega \in B} \left[\sqrt{\left| \omega(z) \right|^2 + \left| \int_0^1 \frac{\omega(tz)}{t^2} dt \right|^2} \right] \leq 2\sqrt{2}. \end{aligned} \quad (22)$$

Inequality (22) follows from the parallelogram law and the last step from (6). And (21) shows that $\mu \leq 1/\sqrt{2}$. $\square$

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SINGH: 3A/95 AZAD NAGAR, KANPUR 208002, INDIA