ON A CLASS OF UNIVALENT FUNCTIONS

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ABSTRACT. We consider the class of univalent functions \( f(z) = z + a_3z^3 + a_4z^4 + \cdots \) analytic in the unit disc and satisfying \(|(z^2f'(z)/f^2(z)) - 1| < 1\), and show that such functions are starlike if they satisfy \(|(z^2f'(z)/f^2(z)) - 1| < (1/\sqrt{2})\).

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Let \( A \) denote the class of functions which are analytic in the unit disc \( U = \{ z : |z| < 1 \} \) and have Taylor series expansion

\[
  f(z) = z + a_2z^2 + a_3z^3 + \cdots ,
\]

and let \( T \) be the univalent [3] subclass of \( A \) which satisfy

\[
  \left| \frac{z^2f'(z)}{f^2(z)} - 1 \right| < 1, \quad z \in U. \tag{2}
\]

By \( T_2 \) we denote the subclass of \( T \) for which \( f''(0) = 0 \). In this paper, we prove the following theorem.

**Theorem 1.** If \( f \in T_2 \), then

1. \( \text{Re}(f(z)/z) > 1/2, \quad z \in U \),
2. \( f \) is starlike in \( |z| < 1/\sqrt{2} = 0.840896\ldots \),
3. \( \text{Re} f'(z) > 0 \) for \( |z| < 1/\sqrt{2} \).

Items (i) and (iii) are improvements of results in [2], and (ii) is the same as in [2] but has a different proof. Furthermore, (i) and (iii) are sharp as shown by the function

\[
  f(z) = \frac{z}{1-z^2},
\]

but the sharpness of (ii) is difficult to establish by a direct example. We also prove the following theorem which partially answers a question raised in [1].

**Theorem 2.** If \( T_{2,\mu} \) is the subclass of \( T_2 \) which satisfies

\[
  \left| \frac{z^2f'(z)}{f^2(z)} - 1 \right| < \mu < 1,
\]

\( T_{2,\mu} \) is a subclass of starlike functions if \( 0 \leq \mu \leq 1/\sqrt{2} \).
We define by $B$ the class of functions $\omega$ analytic in $U$ and satisfying
\[ |\omega(z)| < 1, \quad z \in U, \quad \omega(0) = \omega'(0) = 0. \] (5)
From Schwarz’s lemma it then follows that
\[ |\omega(z)| \leq |z|^2. \] (6)

**Proof of Theorem 1.** If $f \in T_2$ and satisfies (2), then
\[ z^2 \frac{f'(z)}{f^2(z)} - 1 = \omega(z), \quad z \in U, \quad \omega \in B, \] (7)
and by direct integration
\[ \frac{z}{f(z)} = 1 - \int_0^1 \frac{\omega(tz)}{t^2} dt, \quad z \in U, \quad \omega \in B. \] (8)
From (8), we obtain
\[ \left| \frac{z}{f(z)} - 1 \right| \leq |z|^2 < 1, \] (9)
and this gives
\[ \left| 1 - \frac{f(z)}{z} \right| \leq \left| \frac{f(z)}{z} \right|, \] (10)
which is equivalent to $(\text{Re} \ f(z)/z) > 1/2$, This proves (i).
Furthermore, from (9), we obtain
\[ \left| \arg \frac{f(z)}{z} \right| \leq \sin^{-1} |z|^2. \] (11)
From (7), we obtain
\[ z \frac{f'(z)}{f(z)} = f(z) \frac{1 + \omega(z)}{z} \] (12)
and, therefore,
\[ \left| \arg \frac{zf'(z)}{f(z)} \right| = \left| \arg \frac{f(z)}{z} + \arg (1 + \omega(z)) \right| \leq 2 \sin^{-1} |z|^2. \] (13)
This gives (ii).
In order to prove (iii), we notice that (7) yields
\[ f'(z) = \left( \frac{f(z)}{z} \right)^2 (1 + \omega(z)) \] (14)
and, therefore,
\[ \left| \arg f'(z) \right| = \left| 2 \arg \frac{f(z)}{z} + \arg (1 + \omega(z)) \right| \leq 3 \sin^{-1} |z|^2. \] (15)
But this is equivalent to (iii). \(\Box\)

**Proof of Theorem 2.** If $f \in T_{2,\mu}$, we obtain from (4)
\[ z \frac{f'(z)}{f^2(z)} - 1 = \mu \omega(z), \quad \omega \in B, \quad z \in U \quad \text{and} \quad \frac{z}{f(z)} = 1 - \mu \int_0^1 \frac{\omega(tz)}{t^2} dt. \] (16)
Hence
\[ \frac{z \left( f'(z) / f(z) \right)}{1 - \mu} = \frac{1 + \mu \omega(z)}{1 - \mu \int_0^1 (\omega(tz)/t^2) dt} \] \tag{17}

Now \( \text{Re} \left( f'(z) / f(z) \right) > 0 \) is equivalent to the condition
\[ \frac{z \left( f'(z) / f(z) \right)}{1 - \mu} = \frac{1 + \mu \omega(z)}{1 - \mu \int_0^1 (\omega(tz)/t^2) dt} \neq -iT, \quad T \in \mathbb{R}. \] \tag{18}

Relation (18) is equivalent to
\[ \begin{align*}
\mu \left[ \left( \omega(z) + \int_0^1 \frac{\omega(tz)}{t^2} \, dt \right) + \frac{1-iT}{1+iT} \left( \omega(z) - \int_0^1 \frac{\omega(tz)}{t^2} \, dt \right) \right] &\neq -1. \\
\end{align*} \] \tag{19}

Let
\[ M = \sup_{z \in U, \omega \in \mathcal{B}, T \in \mathbb{R}} \left| \left( \omega(z) + \int_0^1 \frac{\omega(tz)}{t^2} \, dt \right) + \frac{1-iT}{1+iT} \left( \omega(z) - \int_0^1 \frac{\omega(tz)}{t^2} \, dt \right) \right|, \] \tag{20}

then, in view of the rotation invariance of \( B \), it follows that
\[ \text{Re} \left( f'(z) / f(z) \right) > 0, \quad \text{if} \ \mu \leq \frac{2}{M}. \] \tag{21}

However, from (20), we notice that
\[ M \leq \sup_{z \in U, \omega \in \mathcal{B}} \left[ \left| \omega(z) \right| + \int_0^1 \left| \frac{\omega(tz)}{t^2} \right| \, dt \right] \leq 2 \sup_{z \in U, \omega \in \mathcal{B}} \left[ \sqrt{\left| \omega(z) \right|^2 + \left| \int_0^1 \frac{\omega(tz)}{t^2} \, dt \right|^2} \right] \leq 2 \sqrt{2}. \] \tag{22}

Inequality (22) follows from the parallelogram law and the last step from (6). And (21) shows that \( \mu \leq 1 / \sqrt{2} \).

**References**


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