MEAN NUMBER OF REAL ZEROS OF A RANDOM HYPERBOLIC POLYNOMIAL

J. ERNEST WILKINS, JR.

(Received 25 March 1998)

ABSTRACT. Consider the random hyperbolic polynomial, \( f(x) = 1^pa_1 \cosh x + \cdots + np \times a_n \cosh nx \), in which \( n \) and \( p \) are integers such that \( n \geq 2, p \geq 0 \), and the coefficients \( a_k(k = 1, 2, \ldots, n) \) are independent, standard normally distributed random variables. If \( \nu_{np} \) is the mean number of real zeros of \( f(x) \), then we prove that \( \nu_{np} = \pi^{-1} \log n + O((\log n)^{1/2}) \).

Keywords and phrases. Random polynomials, real zeros, hyperbolic polynomials, Kac-Rice formula.

2000 Mathematics Subject Classification. Primary 60G99.

1. Introduction. Let \( n \) and \( p \) be integers such that \( n \geq 2 \) and \( p \geq 0 \). We suppose that \( a_k(k = 1, 2, \ldots, n) \) are independent, normally distributed random variables, each with mean 0 and variance 1, and we define the random hyperbolic polynomial \( f(x) \) so that

\[
f(x) = \sum_{k=1}^{n} k^p a_k \cosh kx.
\] (1.1)

We prove the following result.

**THEOREM 1.1.** Let \( \nu_{np} \) be the mean number of real zeros of \( f(x) \). Then

\[
\nu_{np} = \pi^{-1} \log n + O\left\{ (\log n)^{1/2} \right\}.
\] (1.2)

The case when \( p = 0 \) was considered by Das [3], whose result was reported by Bharucha-Reid and Sambandham [1, page 110] in the form \( \nu_{no} \sim \pi^{-1} \log n \). The case when \( p = 1 \) was discussed by Farahmand and Jahangiri [5], who found the result (1.2) in that case.

The principal term in (1.2) is independent of \( p \). That behavior does not occur in the algebraic case [4] (replace \( \cosh kx \) in (1.1) by \( x^k \) and let \( k \) range from 0 to \( n \)), for which \( \nu_{np} \sim \pi^{-1} \left\{ 1 + (2p + 1)^{1/2} \right\} \log n \) (even if \( p \) is a nonnegative real number), and also does not occur in the trigonometric case [2] (replace \( \cosh kx \) in (1.1) by \( \cos kx \) and count zeros on \((0, 2\pi))\), for which \( \nu_{np} = \left\{ (2p + 1)/(2p + 3) \right\}^{1/2} (2n + 1) + O(n^{1/2}) \) (even if \( p \) is a nonnegative real number). The error term in this last case can be replaced by \( O(1) \) when \( 2p \) is a nonnegative integer [6, 7, 8, 9].

2. Preliminary analysis. If we apply the Kac-Rice formula to our problem, we see that

\[
\nu_{np} = \pi^{-1} \int_{-\infty}^{\infty} F_{np}(x) \, dx = 2\pi^{-1} \int_{0}^{\infty} F_{np}(x) \, dx
\] (2.1)
in which

\[ F_{np}(x) = \left( \frac{A_{np}(x)C_{np}(x) - B_{np}^2(x)}{A_{np}(x)} \right)^{1/2}, \tag{2.2} \]

\[ A_{np}(x) = \sum_{k=1}^{n} k^{2p} \cosh^2 kx, \tag{2.3} \]

\[ B_{np}(x) = \sum_{k=1}^{n} k^{2p+1} \sinh kx \cosh kx, \tag{2.4} \]

\[ C_{np}(x) = \sum_{k=1}^{n} k^{2p+2} \sinh^2 kx. \tag{2.5} \]

We furnish explicit formulae for the sums in (2.3), (2.4), and (2.5) in the following lemma.

**Lemma 2.1.** It is true that

\[ 2^{2p+2} A_{np}(x) = (2n+1)^{2p} \csch x \sinh z \]
\[ \times \left[ \sum_{r=0}^{2p} 2pC_r (2n+1)^{-r}\varphi_r(x) + (2n+1)^{-2p} (2^{2p+2} S_{np} - \delta_{op}) \sinh x \csch z \right], \tag{2.6} \]

\[ 2^{2p+3} B_{np}(x) = (2n+1)^{2p+1} \csch x \sinh z \sum_{r=0}^{2p+1} 2p+1C_r (2n+1)^{-r}\varphi_r(x), \tag{2.7} \]

\[ 2^{2p+4} C_{np}(x) = (2n+1)^{2p+2} \csch x \sinh z \]
\[ \times \left[ \sum_{r=0}^{2p+2} 2p+2C_r (2n+1)^{-r}\varphi_r(x) - (2n+1)^{-2p-2} 2^{2p+4} S_{np+1} \sinh x \csch z \right], \tag{2.8} \]

in which

\[ z = (2n+1)x, \tag{2.9} \]

\[ \varphi_{2r}(x) = g_{2r}(x), \quad \varphi_{2r+1}(x) = g_{2r+1}(x) \coth z, \tag{2.10} \]

\[ \psi_{2r}(x) = g_{2r}(x) \coth z, \quad \psi_{2r+1}(x) = g_{2r+1}(x), \tag{2.11} \]

\[ g_r(x) = \sinh x \left\{ \frac{d'(\csch x)}{dx^r} \right\}, \tag{2.12} \]

\[ 2S_{np} = \sum_{k=1}^{n} k^{2p}, \tag{2.13} \]

where \( pC_r \) is the binomial coefficient \( p!/[r!(p-r)!] \), and \( \delta_{op} \) is the Kronecker delta, i.e., \( \delta_{op} = 1 \) when \( p = 0 \) and \( \delta_{op} = 0 \) when \( p \neq 0 \).
With the help of (2.13), the identity $2 \cosh^2 kx = \cosh 2kx + 1$, it is clear that

$$2^{2p+2} A_{np}(x) = \frac{2d^{2p} \left( \sum_{k=1}^{n} (\cosh 2kx + 1) \right)}{dx^{2p}} - 2n \delta_{op} + 2^{2p+2} S_{np}$$

$$= d^{2p} \left[ 4A_{no}(x) \right] - 2n \delta_{op} + 2^{2p+2} S_{np}. \quad (2.14)$$

It is known from [6, equation 2.15] that $4A_{no}(x) = 2n - 1 + \csch x \sinh z$, if $z$ is defined by (2.9). Hence,

$$2^{2p+2} A_{np}(x) = \sum_{r=0}^{2p} 2pC_r \left\{ \frac{d^r (\csch x)}{dx^r} \right\} \left\{ \frac{d^{2p-r} (\sinh z)}{dx^{2p-r}} \right\} - \delta_{op} + 2^{2p+2} S_{np}. \quad (2.15)$$

If the derivatives of $\sinh z$ are calculated and the definitions (2.10) and (2.12) are used, we see that (2.6) is true. In a similar manner, it follows from (2.3), (2.4), and (2.11) that

$$2^{2p+3} B_{np}(x) = \frac{d \left\{ 2^{2p+2} A_{np}(x) \right\}}{dx} = \frac{d^{2p+1} (\csch x \sinh z)}{dx^{2p+1}}$$

$$= (2n + 1)^{2p+1} (\csch x \sinh z) \sum_{r=0}^{2p+1} 2p+1C_r (2n + 1)^{-r} \Psi_r(x), \quad (2.16)$$

so that (2.7) is true. Finally, (2.8) is a consequence of (2.6) and the identity $C_{np}(x) = A_{n,p+1}(x) - 2S_{n,p+1}$.

A straightforward calculation, based on (2.6), (2.7), and (2.8), suffices to prove the following lemma.

**Lemma 2.2.** It is true that

$$2^{4p+6} \left\{ A_{np}(x) C_{np}(x) - B_{np}^2 (x) \right\} = (2n + 1)^{4p+2} \csch^2 x \sinh^2 z$$

$$\times \left[ \sum_{r=0}^{4p+2} (2n + 1)^{-r} \theta_{rp}(x) + \Theta_{np}(x) \sinh x \csch z - \Psi_{np}(x) \sinh^2 (x) \csch^2 z \right]$$

in which

$$\theta_{rp}(x) = \sum_{s=0}^{r} \left\{ 2pC_s \cdot 2p+2C_{r-s} \varphi_s(x) \varphi_{r-s}(x) - 2p+1C_r \cdot 2p+1C_{r-s} \psi_s(x) \psi_{r-s}(x) \right\}, \quad (2.18)$$

$$\Theta_{np}(x) = (2n + 1)^{-2p} (2^{2p+2} S_{np} - \delta_{op}) \sum_{r=0}^{2p+2} 2p+2C_r (2n + 1)^{-r} \varphi_r(x)$$

$$- (2n + 1)^{-2p-2} 2^{2p+4} S_{n,p+1} \sum_{r=0}^{2p} 2pC_r (2n + 1)^{-r} \varphi_r(x), \quad (2.19)$$

$$\Psi_{np}(x) = (2n + 1)^{-4p-2} (2^{2p+2} S_{np} - \delta_{op}) 2^{2p+4} S_{n,p+1}. \quad (2.20)$$

We need the more explicit formulae for $g_r(x)$ contained in the following lemma.
Lemma 2.3. There are constants $\beta_{rs}(s = 0, 1, \ldots, [r/2])$ such that

$$g_{2r}(x) = \sum_{s=0}^{r} \beta_{2r,s} \text{csch}^{2s}x,$$  \hspace{1cm} (2.21)

$$g_{2r+1}(x) = \sum_{s=0}^{r} \beta_{2r+1,s} \text{csch}^{2s}x \coth x.$$  \hspace{1cm} (2.22)

It follows from (2.12) that (2.21) is true when $r = 0$ if $\beta_{00} = 1$. A differentiation of (2.12) shows that

$$g_{r+1}(x) = \frac{d}{dx}g_{r}(x) \coth x.$$  \hspace{1cm} (2.23)

If (2.21) is true for $r$, we infer from (2.23) that (2.22) is true for $r$, provided that

$$\beta_{2r+1,s} = -(2s+1)\beta_{2r,s}. \hspace{1cm} (2.24)$$

Similarly, the truth of (2.21) with $r$ replaced by $r+1$ is assured when

$$\beta_{2r+2,s} = -(2s+1)\beta_{2r+1,s} - 2s\beta_{2r+1,s-1}. \hspace{1cm} (2.25)$$

We record for future reference the cases when $r = 0, 1, 2$:

$$g_0(x) = 1, \quad g_1(x) = -\coth x, \quad g_2(x) = 1 + 2\text{csch}^{2}x.$$  \hspace{1cm} (2.26)

3. Estimates of the terms in (2.6) and (2.17) when $x$ is not too small. We suppose that $x \geq \varepsilon$, in which

$$\varepsilon = \frac{w}{(2n+1)}, \quad w = (\log n)^{1/2}. \hspace{1cm} (3.1)$$

Lemma 3.1. If $n_0 = 8104$ and $n \geq n_0$, the functions $\sinh^3 x \text{csch} z$, $\sinh x \text{csch} z$, and $\sinh^4 x \text{csch}^2 z$ are decreasing functions of $x$ when $x \geq \varepsilon$.

We observe that

$$\frac{\text{csch}^2 x \text{sech} x \sinh^2 z \text{sech} z d(\sinh^3 x \text{csch} z)}{dx} = 3\tanh z - (2n+1) \tanh x < 3 - (2n+1) \tanh \varepsilon.$$  \hspace{1cm} (3.2)

Also,

$$\frac{\cosh^2 \varepsilon d[(2n+1) \tanh \varepsilon]}{dn} = \sinh 2\varepsilon - 2\varepsilon + (2nw)^{-1} > 0. \hspace{1cm} (3.3)$$

Therefore, $(2n+1) \tanh \varepsilon > 3$ when $n \geq n_0$ because $(2n+1) \tanh \varepsilon > 3$ when $n = 8104$. It follows that $\sinh^3 x \text{csch} z$ is decreasing when $x \geq \varepsilon$ and $n \geq n_0$. The other functions in the lemma are decreasing because $(\sinh^3 x \text{csch} z)^{1/3} \text{csch}^{2/3} z$ and $(\sinh^3 x \text{csch} z)^{4/3} \text{csch}^{2/3} z$ are. The third term on the right hand side of (2.17) is estimated in the following lemma.
Lemma 3.2. When \( n \geq n_0 \) and \( x \geq \varepsilon \), it is true that
\[
\Psi_{np}(x) \sinh^2 x \operatorname{csch}^2 z = O\left( w^4 e^{-2w} \right)(2n + 1)^{-2} \operatorname{csch}^2 x.
\]
(3.4)

It follows from an explicit formula [6, equation (2.12)] for \( S_{np} \) that \( S_{np} = O\{(2n + 1)^{2p+1}\} \). Then (2.20) and Lemma 3.1 imply that
\[
\Psi_{np}(x) \sinh^4 x \operatorname{csch}^2 z = O\left\{ (2n + 1)^2 \sinh^4 \varepsilon \operatorname{csch}^2 w \right\}
\]
(3.5)
\[
\begin{align*}
&= O\left\{ (2n + 1)^2 \varepsilon^4 e^{-2w} \right\}. \\
\end{align*}
\]

Lemma 3.2 is an immediate consequence of this result and (3.1).

Lemma 3.3. When \( x \geq \varepsilon \), it is true that \( g_r(x) = O(\varepsilon^{-r}) \), \( \varphi_r(x) = O(\varepsilon^{-r}) \), and \( \psi_r(x) = O(\varepsilon^{-r}) \).

The lemma follows immediately from (2.10), (2.11), (2.21), and (2.22), and the facts that
\[
\operatorname{csch} x \leq \operatorname{csch} \varepsilon < \varepsilon^{-1},
\]
\[
\operatorname{coth} x \leq \operatorname{coth} \varepsilon < \varepsilon^{-1} \operatorname{cosh} \varepsilon_0,
\]
\[
\operatorname{coth} z \leq \operatorname{coth} w \leq \operatorname{coth} w_0,
\]
in which \( \varepsilon_0 = w_0/(2n_0 + 1) \) and \( w_0 = (\log n_0)^{1/2} \). Now, we can estimate the second term on the right-hand side of (2.17).

Lemma 3.4. When \( n \geq n_0 \) and \( x \geq \varepsilon \), it is true that
\[
\Theta_{np}(x) \sinh x \operatorname{csch} z = O\left( w^3 e^{-w} \right)(2n + 1)^{-2} \operatorname{csch}^2 x.
\]
(3.7)

We deduce from (2.19), Lemmas 3.1 and 3.3, and the earlier observation that \( S_{np} = O\{(2n + 1)^{2p+1}\} \) that
\[
\Theta_{np}(x) \sinh^3 x \operatorname{csch} z = O\left\{ (2n + 1)^2 \sum_{r=0}^{2p+2} O\left(w^{-r}\right) \sinh^3 \varepsilon \operatorname{csch} w \right\}
\]
(3.8)
\[
\begin{align*}
&= O\left\{ (2n + 1)^2 \varepsilon^3 e-w \right\} = O\left(w^3 e^{-w} \right)(2n + 1)^{-2}.
\end{align*}
\]

This equation suffices to prove Lemma 3.4.

The analysis to obtain an estimate for \( \theta_{rp} \) is more recondite. We use (2.10), (2.11), (2.18), and the identity \( \operatorname{coth}^2 z = 1 + \operatorname{csch}^2 z \), to see that
\[
\theta_{2r,p} = \sum_{s=0}^{2r} L_{2r,s,p} g_s(x) g_{2r-s}(x) + M_{rp}(x) \operatorname{csch}^2 z,
\]
(3.9)
in which
\[
L_{rs} = 2 p C_s 2 p + 2 C_{r-s} - 2 p + 1 C_s 2 p + 1 C_{r-s},
\]
(3.10)
\[
M_{rp}(x) = \sum_{s=0}^{r-1} 2 p C_{2s+1} 2 p + 2 C_{2r-2s-1} g_{2s+1}(x) g_{2r-2s-1}(x)
\]
\[
- \sum_{s=0}^{r} 2 p + 1 C_{2s} 2 p + 1 C_{2r-2s} g_{2s}(x) g_{2r-2s}(x).
\]
(3.11)
In a similar manner, we also see that

$$\theta_{2r+1,p}(x) = \sum_{s=0}^{2r+1} L_{2r+1,sp} g_s(x) g_{2r+1-s}(x) \coth z.$$  \hfill (3.12)

Because we infer from (3.9) and Lemma 3.3 that $M_{rp}(x) = O(\epsilon^{-2r})$, it follows, from (3.9) and (3.12), that

$$\theta_{rp}(x) = \sum_{s=0}^{r} L_{rsp} g_s(x) g_{r-s}(x) (\coth z)^{ur} + O(\epsilon^{-2r}) \csch^2 z$$ \hfill (3.13)

in which $u_r = \{1 - (-1)^r\}/2$. Moreover, Lemma 2.3 implies that

$$g_r(x) = \sum_{h=0}^{[r/2]} \beta_{rh} \csch^{2h} x (\coth z)^{ur},$$ \hfill (3.14)

so that there are constants $y_{rsh}$ such that

$$g_s(x) g_{r-s}(x) = \sum_{h=0}^{[r/2]} y_{rsh} \csch^{2h} x (\coth x)^{ur}. \hfill (3.15)$$

In the derivation of (3.15), it is helpful to consider separately the cases when $r$ is even and $s$ is odd. When $r$ is even and $s$ is odd, we also need the identity $\coth^2 x = 1 + \csch^2 x$. An easy induction using (2.24) and (2.25) when $s = 0$ shows that $\beta_{ro} = (-1)^r$; hence $Y_{rso} = (-1)^r$.

The combinatorial identity

$$\sum_{s=0}^{r} L_{rsp} = 0 \hfill (3.16)$$

is well known (and is easy to prove). We now deduce, from (3.13), (3.15), and (3.16), that

$$\theta_{rp}(x) \sinh^2 x = \sum_{s=0}^{r} L_{rsp} \sum_{h=1}^{[r/2]} y_{rsh} \csch^{2h-2} x (\coth x \coth z)^{ur}$$

$$+ O(\epsilon^{-r}) \sinh^2 x \csch^2 z.$$ \hfill (3.17)

We showed in the proof of Lemma 3.3 that

$$\csch x = O(\epsilon^{-1}), \quad \coth x = O(\epsilon^{-1}), \quad \coth z = O(1). \hfill (3.18)$$

Because it follows from Lemma 2.3 that

$$\sinh^2 x \csch^2 z \leq \sinh^2 \epsilon \csch^2 w = O(\epsilon^2 e^{-2w}),$$ \hfill (3.19)

we conclude that the following lemma is true.

**Lemma 3.5.** When $n \geq n_o$ and $x \geq \epsilon$, it is true that

$$\theta_{rp}(x) = O(\epsilon^{2-r}) \{1 + O(e^{-2w})\} \csch^2 x.$$ \hfill (3.20)
We also need the more precise estimates of $\theta_{rp}(x)$ when $r = 0, 1, \text{ and } 2$, deductible from (2.10), (2.11), (2.18), and (2.26), that are recorded below:

\[
\begin{align*}
\theta_{0p}(x) &= -\csch^2 z = O(w^2 e^{-2w})(2n+1)^{-2} \csch^2 x, \\
\theta_{1p}(x) &= 0, \\
\theta_{2p}(x) &= (1 - 4p^2 \csch^2 z + 2p \sinh x \csch z) \csch^2 x \\
&= [1 + O(e^{-2w}) + (2n+1)^{-1}O(w e^{-w})] \csch^2 x \\
&= [1 + O(e^{-2w})] \csch^2 x.
\end{align*}
\]

(3.21)

Finally, the methods used above can be applied to (2.6) to yield an easy proof of the following lemma.

**Lemma 3.6.** When $n \geq n_0$ and $x \geq \varepsilon$, it is true that

\[
2^{4p+2} A_{np} = (2n+1)^{2p} \csch x \sinh z \left[ 1 + O(w^{-1}) \right].
\]

(3.22)

**Proof of Theorem 1.1.** If we use Lemmas 3.2, 3.4, and 3.5, we infer from (2.17), and (3.21) that, when $n \geq n_0$ and $x \geq \varepsilon$,

\[
2^{4p+6} \left[ A_{np}(x) C_{np}(x) - B_{np}^2(x) \right] = (2n+1)^{4p} \csch^4 x \sinh^2 z \left[ 1 + O(w^{-1}) \right].
\]

(3.23)

It now follows from (2.2) and Lemma 3.6 that, when $n \geq n_0$ and $x \geq \varepsilon$,

\[
\begin{align*}
2F_{np}(x) \, dx &= [1 + O(w^{-1})] \csch x, \\
2 \int_{\varepsilon}^{\infty} F_{np}(x) \, dx &= \left[ 1 + O(w^{-1}) \right] \log \left\{ \coth \left( \frac{\varepsilon}{2} \right) \right\} \\
&= \left[ 1 + O(w^{-1}) \right] \left[ 1 + O(w^{-2} \log w) \right] \log n, \\
2\pi^{-1} \int_{\varepsilon}^{\infty} F_{np}(x) \, dx &= \pi^{-1} \log n + O \left\{ (\log n)^{1/2} \right\}.
\end{align*}
\]

(3.24) \hspace{1cm} (3.25) \hspace{1cm} (3.26)

Next, we observe that (2.2), (2.3), and (2.5) imply that

\[
0 \leq C_{np}(x) \leq n^2 \sum_{k=1}^{n} k^{2p} \sinh^2 kx < n^2 A_{np}(x),
\]

(3.27)

\[
0 \leq F_{np}(x) \leq \left\{ \frac{C_{np}(x)}{A_{np}(x)} \right\}^{1/2} < n,
\]

(3.28)

\[
2\pi^{-1} \int_{0}^{\varepsilon} F_{np}(x) \, dx < 2\pi^{-1} n \varepsilon < \pi^{-1} e = O \left\{ (\log n)^{1/2} \right\}.
\]

(3.29)

If we add (3.26) and (3.29) and use (2.1), we see that the theorem is true. \hfill \Box

**References**


WILKINS: DEPARTMENT OF MATHEMATICS, CLARK ATLANTA UNIVERSITY, ATLANTA, GA 30314, USA