A GENERAL EXISTENCE PRINCIPLE FOR FIXED POINT THEOREMS IN D -METRIC SPACES

B. C. DHAGE, A. M. PATHAN, and B. E. RHOADES

(Received 13 April 1998 and in revised form 22 July 1998)

ABSTRACT. We establish two general principles for fixed point theorems in D -metric spaces, and then show that several theorems in D -metric spaces follow as corollaries of these general principles.

Keywords and phrases. α -condensing maps, D -metric spaces, fixed point theorems.

2000 Mathematics Subject Classification. Primary 47H10.

1. Introduction. The concept of a D -metric space was introduced by the first author in [1]. A nonempty set X , together with a function $D : X \times X \times X \rightarrow [0, \infty)$ is called a D -metric space, denoted by (X, D) if D satisfies

- (i) $D(x, y, z) = 0$ if and only if $x = y = z$ (coincidence),
- (ii) $D(x, y, z) = D(p\{x, y, z\})$, where p is a permutation of x, y, z (symmetry),
- (iii) $D(x, y, z) \leq D(x, y, a) + D(x, a, z) + D(a, y, z)$ for all $x, y, z, a \in X$ (tetrahedral inequality).

The nonnegative real function D is called a D -metric on X . Some specific examples of D -metrics appear in [2]. A D -metric is a continuous function on X^3 in the topology of D -metric convergence, which is Hausdorff (see [5]).

In this paper, we establish two general fixed point principles for mappings in a D -metric space, which yield several fixed point theorems as corollaries.

2. Preliminaries. Let $f : X \rightarrow X$. The orbit of f at the point $x \in X$ is the set $O(x) = \{x, fx, f^2x, \dots\}$. An orbit of x is said to be bounded if there exists a constant $K > 0$ such that $D(u, v, w) \leq K$ for all $u, v, w \in O(x)$. The constant K is called a D -bound of $O(x)$. A D -metric space X is said to be f -orbitally bounded if $O(x)$ is bounded for each $x \in X$. A sequence $x_n \subset X$ is said to be D -Cauchy if, for each $\varepsilon > 0$, there exists a positive integer n_0 such that, for all $m > n, p \geq n_0, D(x_m, x_n, x_p) < \varepsilon$. A sequence $\{x_n\} \subset X$ is said to be D -convergent to a point $x \in X$ if, for each $\varepsilon > 0$ there exists a positive integer n_0 such that, for all $m, n \geq n_0, D(x_m, x_n, x) < \varepsilon$. An orbit $O(x)$ is called f -orbitally complete if every D -Cauchy sequence in $O(x)$ converges to a point in X .

LEMMA 2.1 [4]. *Let $\{x_n\} \subset X$ be a bounded sequence with D -bound K satisfying*

$$D(x_n, x_{n+1}, x_m) \leq \lambda^n K \tag{2.1}$$

for all positive integers $m > n$, and some $0 \leq \lambda < 1$. Then $\{x_n\}$ is D -Cauchy.

3. Main results

THEOREM 3.1. *Let (X, D) be a D -metric spaces, f a selfmap of X . Suppose that there exists an $x_0 \in X$ such that $O(x_0)$ is D -bounded and f -orbitally complete. Suppose also that f satisfies*

$$D(fx, fy, fz) \leq \lambda \max \{D(x, y, z), D(x, fx, z)\} \quad \text{for } x, y, z \in \overline{O(x_0)} \quad (3.1)$$

for some $0 \leq \lambda < 1$. Then f has a unique fixed point in X .

PROOF. Suppose there exists an n such that $x_n = x_{n+1}$. Then f has x_n as a fixed point in X . Therefore we may assume that all of the x_n are distinct.

We wish to show that, for any positive integers m, n , $m > n$, that $D(x_{n+1}, x_{n+2}, x_m) \leq \lambda^n K$, where K is the D -bound of $O(x_0)$. The proof is by induction. From (3.1), for any m ,

$$D(x_1, x_2, x_m) \leq \lambda \max \{D(x_0, x_1, x_{m-1}), D(x_0, x_1, x_{m-1})\} \leq \lambda K. \quad (3.2)$$

Again using (3.1),

$$D(x_2, x_3, x_m) \leq \lambda \max \{D(x_2, x_3, x_{m-1}), D(x_1, x_2, x_{m-1})\}. \quad (3.3)$$

Using (3.2),

$$D(x_2, x_3, x_m) \leq \lambda \max \{D(x_2, x_3, x_{m-1}), \lambda K\}. \quad (3.4)$$

Inequality (3.4) can be regarded as a recursion formula in m . Therefore

$$D(x_2, x_3, x_m) \leq \lambda \max \{\lambda \max \{D(x_2, x_3, x_{m-2}), \lambda K\}, \lambda K\} \leq \lambda^2 K. \quad (3.5)$$

Assume the induction hypothesis. Then, from (3.1),

$$\begin{aligned} D(x_{n+1}, x_{n+2}, x_m) &\leq \lambda \max \{D(x_{n+1}, x_{n+2}, x_{m-1}), D(x_n, x_{n+1}, x_{m-1})\} \\ &\leq \lambda \max \{D(x_{n+1}, x_{n+2}, x_{m-1}), \lambda^n K\}. \end{aligned} \quad (3.6)$$

Inequality (3.6) can be regarded as a recursion formula in m . Therefore,

$$\begin{aligned} D(x_{n+1}, x_{n+2}, x_m) &\leq \lambda \max \{\lambda \max \{D(x_{n+1}, x_{n+2}, x_{m-2}), \lambda^n K\}, \lambda^n K\} \\ &= \max \{\lambda^2 D(x_{n+1}, x_{n+2}, x_{m-2}), \lambda^{n+2} K, \lambda^{n+1} K\} \\ &= \max \{\lambda^2 D(x_{n+1}, x_{n+2}, x_{m-2}), \lambda^{n+1} K\} \\ &\leq \max \{\lambda^2 \cdot \lambda \max \{D(x_{n+1}, x_{n+2}, x_{m-3}), \lambda^{n+1} K\}, \lambda^{n+1} K\} \\ &= \max \{\lambda^3 D(x_{n+1}, x_{n+2}, x_{m-3}), \lambda^{n+1} K\} \leq \dots \\ &\leq \max \{\lambda^n D(x_{n+1}, x_{n+2}, x_{m-n}), \lambda^{n+1} K\} \\ &\leq \max \{\lambda^n \cdot \lambda \max \{D(x_{n+1}, x_{n+2}, x_{m-n-1}), \lambda^{n+1} K\}, \lambda^{n+1} K\} \\ &= \lambda^{n+1} K, \end{aligned} \quad (3.7)$$

and $\{x_n\}$ is D -Cauchy by Lemma 2.1. Since X is x_0 -orbitally complete, there exists a $p \in X$ with $\lim x_n = p$.

In (3.1) set $x = x_n$, $z = p$ to obtain

$$D(x_{n+1}, x_{n+1}, fp) \leq \lambda \max \{D(x_n, x_n, p), D(x_n, x_{n+1}, p)\}. \quad (3.8)$$

Taking the limit of (3.8) as $n \rightarrow \infty$ yields $D(p, p, fp) \leq \lambda D(p, p, p) = 0$, and $p = fp$. To prove uniqueness, suppose that q is also a fixed point of f . Then, from (3.1),

$$D(p, p, q) = D(fp, fp, fq) \leq \lambda \max \{D(p, p, q), D(p, fp, q)\} = \lambda D(p, p, q), \quad (3.9)$$

which implies that $p = q$. $\square$

COROLLARY 3.2 [2, Theorem 2.1]. *Let f be a selfmap of a complete and bounded D -metric space X satisfying*

$$D(fx, fy, fz) \leq \lambda D(x, y, z) \quad (3.10)$$

for all $x, y, z \in X$, for some $0 \leq \lambda < 1$. Then f has a unique fixed point p , and f is continuous at p .

PROOF. In (3.10) set $y = fx$ to obtain (3.1). Then, from Theorem 3.1, f has a unique fixed point p .

To prove continuity, let $\{z_n\} \subset X$ with $\lim z_n = p$. From (3.10),

$$D(p, p, fz_n) = D(fp, fp, fz_n) \leq \lambda D(p, p, z_n). \quad (3.11)$$

Taking the limit as $n \rightarrow \infty$ gives $\limsup D(p, p, fz_n) = 0$, and $\liminf D(p, p, fz_n) = 0$ which implies that $\lim fz_n = p = fp$, and f is continuous at p . $\square$

COROLLARY 3.3 [2, Corollary 1.1]. *Let f be a selfmap of a complete and bounded D -metric space satisfying the condition that there exists a positive integer q such that*

$$D(f^q x, f^q y, f^q z) \leq \lambda D(x, y, z) \quad (3.12)$$

for all $x, y, z \in X$, for some $0 \leq \lambda < 1$. Then f has a unique fixed point p , and f is f -orbitally continuous at p .

PROOF. Define $T = f^q$. Then (3.12) reduces to (3.10), and T has a unique fixed point p by Corollary 3.2; i.e., $p = Tp = f^q p$. Thus $fp = f^{q+1} p = T(fp)$, and fp is also a fixed point of T . Uniqueness implies that $fp = p$, and p is a fixed point of f . Condition (3.12) implies the uniqueness of p as a fixed point of f .

For the continuity, let $\{z_n\} \subset O(f)$, with $\lim z_n = p$. From (3.12),

$$D(f^q p, f^q p, f^q z_n) \leq \lambda D(p, p, z_n). \quad (3.13)$$

Taking the limit as $n \rightarrow \infty$ shows that $\lim f^q z_n = p = f^q p$, and f^q is f -orbitally continuous at p . But, since each $z_n \in O(f)$, $\lim f^q z_n = \lim fz_{n+q-1}$, and f is f -orbitally continuous at p . $\square$

COROLLARY 3.4. *Let f be a selfmap of X , X an f -orbitally bounded and complete D -metric space satisfying*

$$D(fx, fy, fz) \leq \alpha \left[\frac{1 + D(x, fx, z)}{1 + D(x, y, z)} \right] D(y, fy, z) + \beta D(x, y, z) \quad (3.14)$$

for all $x, y, z \in X$, $\alpha, \beta \geq 0$, $\alpha + \beta < 1$. Then f has a unique fixed point p and f is continuous at p .

PROOF. In (3.14) set $y = fx$ to obtain

$$\begin{aligned} D(fx, f^2x, fz) &\leq \alpha D(fx, f^2x, z) + \beta D(x, fx, z) \\ &\leq \lambda \max \{D(fx, f^2x, z), D(x, fx, z)\}, \end{aligned} \quad (3.15)$$

where $\lambda = \alpha + \beta < 1$, and (3.1) is satisfied. The conclusion follows from Theorem 3.1.

To prove the continuity of f at p , let $\{z_n\} \subset X$ with $\lim z_n = p$. In (3.14) set $x = z = p$, $y = z_n$, to obtain

$$\begin{aligned} D(p, fz_n, p) &= D(fp, fz_n, fp) \\ &\leq \alpha \left[\frac{1 + D(p, fp, p)}{1 + D(p, z_n, p)} \right] D(z_n, fz_n, p) + \beta D(p, z_n, p) \\ &\leq \alpha D(z_n, fz_n, p) + \beta D(p, z_n, p). \end{aligned} \quad (3.16)$$

Taking the limsup of both sides of (3.16) as $n \rightarrow \infty$ yields

$$D(p, \limsup fz_n, p) \leq \alpha D(p, \limsup fz_n, p), \quad (3.17)$$

which implies that $\limsup fz_n = p$. Similarly, taking the liminf of both sides of (3.16) as $n \rightarrow \infty$ yields

$$D(p, \liminf fz_n, p) \leq \alpha D(p, \liminf fz_n, p), \quad (3.18)$$

which implies that $\liminf fz_n = p$. Therefore $\lim fz_n = p = fp$, and f is continuous at p . $\square$

COROLLARY 3.5. Let f be a selfmap of an f -orbitally bounded and complete D -metric space X , q a fixed positive integer. Suppose that f satisfies

$$D(f^q x, f^q y, f^q z) \leq \alpha \left[\frac{1 + D(x, f^q x, z)}{1 + D(x, y, z)} \right] D(y, f^q y, z) + \beta D(x, y, z) \quad (3.19)$$

for all $x, y, z \in X$, where $\alpha, \beta \geq 0$, $\alpha + \beta < 1$. Then f has a unique fixed point p and f is f -orbitally continuous at p .

PROOF. Set $T = f^q$. Then T satisfies (3.14). Therefore T has a unique fixed point at p , and is continuous at p . A standard argument then verifies that f has p as a unique fixed point. As in the proof of Corollary 3.3, f is f -orbitally continuous at p . $\square$

4. α -condensing maps. For any set A in a D -metric space X , the D -diameter of A , $\delta(A)$, is defined by $\delta(A) = \sup_{x, y, z \in A} D(x, y, z)$. The measure of noncompactness of a bounded set A in a D -metric space X is a nonnegative real number $\alpha(A)$ defined by

$$\alpha(A) = \inf \{ \gamma > 0 : A = \cup_{i=1}^n A_i \text{ for which } \delta(A_i) \leq \gamma \text{ for } i = 1, 2, \dots, n \}. \quad (4.1)$$

DEFINITION 4.1. A selfmap f of X is called α -condensing if, for any bounded set A in X , $f(A)$ is bounded and $\alpha(f(A)) < \alpha(A)$ if $\alpha(A) > 0$.

Some authors refer to α -condensing maps as densifying maps.

LEMMA 4.2. Let $f : X \rightarrow X$, X an f -orbitally bounded and complete D -metric space, be α -condensing. Then $\overline{O(x)}$ is compact for each $x \in X$.

PROOF. Let $x \in X$ and define $A \subset X$ by $A = \{x_n\}$, where $x_n = f^n x$. Then

$$A = \{x, fx, f^2x, \dots\} = \{x\} \cup \{fx, f^2x, \dots\} = \{x\} \cup f(A). \quad (4.2)$$

Therefore, if A is not precompact, then $\alpha(A) = \alpha(f(A)) < \alpha(A)$, a contradiction. Therefore $\bar{A} = \overline{O(x)}$ is compact, since $\bar{A}$ is a complete D -metric space. $\square$

Define $\delta(x, y, z) = \delta(O(x) \cup O(y) \cup O(z))$

THEOREM 4.3. Let f be a continuous compact selfmap of a bounded D -metric space X , satisfying

$$D(f^r x, f^s y, f^t z) < \delta(x, y, z) \text{ for each } x, y, z \in X, \text{ with two of } \{x, y, z\} \text{ distinct,} \quad (4.3)$$

where r, s , and t are fixed positive integers. Then f has a unique fixed point in X .

PROOF. Since f is compact, there exists a compact subset Y of X containing fX . Then $fY \subset Y$ and $A := \bigcap_{n=1}^{\infty} f^n Y$ is a nonempty compact f -invariant subset of X which is mapped by f onto itself. A has the same properties with respect to f^r, f^s , and f^t .

Suppose that $\delta(A) > 0$. Since A is compact there exist $x, y, z \in A$ such that $\delta(A) = D(x, y, z)$. Since $fA = A$, there exist x', y' , and z' in A such that $x = f^r x'$, $y = f^s y'$, and $z = f^t z'$. Then, from (4.3),

$$\delta(A) = D(x, y, z) = D(f^r x', f^s y', f^t z') < \delta(x, y, z) = \delta(A), \quad (4.4)$$

a contradiction. Therefore A consists of a single point, which is a fixed point of f .

Suppose p and q are fixed points of f , $p \neq q$. Then, from (4.3),

$$0 < D(p, p, q) = D(f^r p, f^s p, f^t q) < D(p, p, q), \quad (4.5)$$

a contradiction. Therefore the fixed point is unique. $\square$

COROLLARY 4.4 [8, Theorem 2]. Let X be a compact D -metric space, f a continuous selfmap of X satisfying

$$\begin{aligned} & D(fx, fy, fz) \\ & < \max \{D(x, y, z), D(x, fx, z), D(y, fy, z), D(x, fy, z), D(y, fx, z)\} D(p, p, q) \end{aligned} \quad (4.6)$$

for all $x, y, z \in X$ with $x \neq fx$, $y \neq fy$, or $z \neq fz$. Then f has a unique fixed point p in X .

PROOF. Inequality (4.6) implies that $D(fx, fy, fz) < \delta(x, y, z)$, and the existence and uniqueness of a fixed point p follows from Theorem 4.3.

For continuity, let $\{z_n\} \subset X$ with $z_n \neq p$ for each n and $\lim z_n = p$. From (4.6)

$$D(p, p, fz_n) = D(fp, fp, fz_n) < D(p, fp, z_n). \quad (4.7)$$

Taking the limit as $n \rightarrow \infty$ implies that f is continuous at p . $\square$

THEOREM 4.5. *Let f be an f -orbitally continuous α -condensing selfmap of a complete bounded D -metric space X . Let $a \in X$. If (4.3) holds on $\overline{O(a)}$, then f has a unique fixed point $p \in \overline{O(a)}$, and $\lim_n f^n x = p$ for each $x \in \overline{O(a)}$.*

PROOF. From Lemma 4.2, $\overline{O(a)}$ is compact. Since f is a continuous α -condensing selfmap of $\overline{O(a)}$, f is compact. Now apply Theorem 4.3. $\square$

COROLLARY 4.6. *Let f be a continuous α -condensing selfmap of a complete bounded D -metric space X satisfying (4.6) for all $x, y, z \in X$ with $x \neq fx$, $y \neq fy$, or $z \neq fz$. Then f has a unique fixed point p in X .*

As in the proof of Corollary 4.4, $D(fx, fy, fz) < \delta(x, y, z)$ and the result follows from Theorem 4.5.

THEOREM 4.7. *Let f be a selfmap of a D -metric space X . Suppose that there exists a point $a \in X$ with $\overline{O(a)}$ bounded and complete. Suppose that f is continuous and α -condensing on $\overline{O(a)}$ and satisfies (4.3) for each $x, y, z \in \overline{O(a)}$ with two of $\{x, y, z\}$ distinct, and $x \neq fx$, $y \neq fy$, $z \neq fz$. Then f has a fixed point in $\overline{O(a)}$.*

PROOF. By Lemma 4.2 $\overline{O(a)}$ is compact. If there exists some integer n for which $f^n a = f^{n+1} a$, then f has a fixed point in $\overline{O(a)}$. Assume that $f^n a \neq f^{n+1} a$ for each n . Note that f , restricted to $\overline{O(a)}$ is a continuous compact selfmap of $\overline{O(a)}$. Suppose that $u \neq fu$ for each cluster point u of $\overline{O(a)}$. Then f satisfies condition (4.3) for all $x, y, z \in \overline{O(a)}$, with two of $\{x, y, z\}$ distinct. Therefore, by Theorem 4.3, f , restricted to $\overline{O(a)}$, has a unique fixed point $p \in \overline{O(a)}$. This contradicts the assumption that $u \neq fu$ for each cluster point u of $\overline{O(a)}$. Therefore $u = fu$ for some cluster point $u \in \overline{O(a)}$.

The proofs of Theorems 4.3, 4.5, and 4.7 are very similar to their metric space counterparts in [6] and [7], but have been given here for completeness.

The following results are proved using the proof technique analogous to the corresponding metric space theorems. $\square$

THEOREM 4.8. *Let f be a selfmap of X , an f -orbitally bounded and complete D -metric space. Suppose that f is α -condensing, f -orbitally continuous and satisfies*

$$D(fx, fy, fz) < \alpha \left[\frac{1 + D(x, fx, z)}{1 + D(x, y, z)} \right] D(y, fy, z) + \beta D(x, y, z) = M(x, y, z) \quad (4.8)$$

for all $x, y, z \in X$ with $x \neq fx$, $y \neq fy$, $z \neq fz$, where $\alpha, \beta > 0$, $\alpha + \beta \leq 1$. Then f has a unique fixed point $p \in X$ and f is continuous at p .

PROOF. If $\alpha + \beta < 1$, the result follows from Corollary 3.4. Therefore we assume that $\alpha + \beta = 1$. Let $x_0 \in X$ and define $x_{n+1} = fx_n$, $n \geq 0$. From Lemma 4.2 it follows that $\overline{O(x_0)}$ is compact. Obviously $f : \overline{O(x_0)} \rightarrow \overline{O(x_0)}$.

CASE I. There exists some $x, y, z \in \overline{O(x_0)}$ for which $M = 0$. Then $y = fy = z = x$, and y is a fixed point of f . Inequality (4.8) implies uniqueness.

CASE II. $M \neq 0$ for all $x, y, z \in \overline{O(x_0)}$. Define a function $F : (\overline{O(x_0)})^3 \rightarrow [0, \infty)$ by

$$F(x, y, z) = \frac{D(fx, fy, fz)}{M(x, y, z)}. \quad (4.9)$$

The function F is well defined on $(\overline{O(x_0)})^3$ since $M \neq 0$ on $\overline{O(x_0)}$.

Since F is continuous on $\overline{O(x_0)}$, it attains its maximum value at some point $(u, v, w) \in \overline{O(x_0)}$. We call this maximum value c . From (4.8) it follows that $0 < c < 1$. Therefore

$$\begin{aligned} D(fx, fy, fz) &\leq cM(x, y, z) \\ &\leq \alpha' \left[\frac{1 + D(x, fx, z)}{1 + D(x, y, z)} \right] D(y, fy, z) + \beta' D(x, y, z) \end{aligned} \quad (4.10)$$

for all $x, y, z \in \overline{O(x_0)}$, where $\alpha' = c\alpha > 0$, $\beta' = c\beta > 0$, and $\alpha' + \beta' = c(\alpha + \beta) < 1$. Since $\overline{O(x_0)}$ is compact, it is bounded and complete. The result follows from Corollary 3.4. $\square$

COROLLARY 4.9. *Let f be a selfmap of a complete and f -orbitally bounded D -metric space. Suppose that f is α -condensing and f -orbitally continuous. Let q be a positive integer. Suppose that f satisfies*

$$D(f^q x, f^q y, f^q z) < \alpha \left[\frac{1 + D(x, f^q x, z)}{1 + D(x, y, z)} \right] D(y, f^q y, z) + \beta D(x, y, z) \quad (4.11)$$

for all $x, y, z \in X$ for which the right-hand side of (4.11) is not zero, where $\alpha, \beta > 0$, $\alpha + \beta \leq 1$. Then f has a unique fixed point p and f is f -orbitally continuous at p .

PROOF. Set $T = f^q$. Then T satisfies (4.8), and the existence and uniqueness of the fixed point p , for T , follows from Theorem 4.8. It then follows that p is the unique fixed point for f . The continuity argument is the same as that used in the proof of Corollary 3.3. $\square$

COROLLARY 4.10. *Let f be a continuous selfmap of a compact D -metric space satisfying (4.8). Then f has a unique fixed point p , and f is continuous at p .*

This result is an immediate consequence of Theorem 4.8.

Corollary 4.10 includes [3, Theorem 2.2] as a special case.

REFERENCES

- [1] B. C. Dhage, *A study of some fixed point theorems*, Ph.D. thesis, Marathwada University, Aurangabad, India, 1984.
- [2] ———, *Generalised metric spaces and mappings with fixed point*, Bull. Calcutta Math. Soc. **84** (1992), no. 4, 329–336. MR 93k:54075. Zbl 782.54037.
- [3] ———, *On continuity of mappings in D -metric spaces*, Bull. Calcutta Math. Soc. **86** (1994), no. 6, 503–508. CMP 1 352 051. Zbl 836.54006.
- [4] ———, *On Kanan type maps in D -metric spaces*, J. Natur. Phys. Sci. **11** (1997), 21–38. CMP 1 659 334.
- [5] ———, *Generalized metric spaces and topological structure, I*, An. Științ. Univ. Al. I. Cuza Iași. Mat. (N.S.) (1998), to appear.

- [6] S. Park, *On studying maps of metric spaces*, Honam J. **1** (1981), 23–30.
- [7] S. Park and B. E. Rhoades, *Extensions of some fixed point theorems of Fisher and Janos*, Bull. Acad. Polon. Sci. Sér. Sci. Math. **30** (1982), no. 3-4, 167–169. MR 83j:54046. Zbl 503.54050.
- [8] B. E. Rhoades, *A fixed point theorem for generalized metric spaces*, Internat. J. Math. Math. Sci. **19** (1996), no. 3, 457–460. CMP 1 386 544. Zbl 857.54044.

DHAGE AND PATHAN: MATHEMATICAL RESEARCH CENTRE, MAHATMA GANDHI MAHAVIDYALAYA, AHMEDPUR-413 515, INDIA

RHOADES: DEPARTMENT OF MATHEMATICS, INDIANA UNIVERSITY, BLOOMINGTON, INDIANA 47405-4301, USA

E-mail address: rhoades@indiana.edu