MATRIX-VARIATE BETA DISTRIBUTION

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ABSTRACT. We propose matrix-variate beta type III distribution. Several properties of this distribution including Laplace transform, marginal distribution and its relationship with matrix-variate beta type I and type II distributions are also studied.

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1. Introduction. The random variable u with the probability density function (pdf)

$$(\beta(a, b))^{-1} u^{a-1} (1-u)^{b-1}, \quad 0 < u < 1, \quad (1.1)$$

where $a > 0$ and $b > 0$, is said to have a beta type I distribution with parameters (a, b) . The random variable v with pdf,

$$(\beta(a, b))^{-1} v^{a-1} (1+v)^{-(a+b)}, \quad v > 0, \quad (1.2)$$

where $a > 0$ and $b > 0$, is said to have beta type II distribution with parameters (a, b) . Since (1.2) can be obtained from (1.1) by the transformation $v = u/(1-u)$, some authors call the distribution of v an “inverted beta distribution.”

The matrix variate generalizations of (1.1) and (1.2) are given as follows (see [1, 3, 4, 6, 11]).

DEFINITION 1.1. A $p \times p$ random symmetric positive definite matrix U is said to have a matrix-variate beta type I distribution with parameters (a, b) , denoted as $U \sim B_p^I(a, b)$, if its pdf is given by

$$(\beta_p(a, b))^{-1} \det(U)^{a-(p+1)/2} \det(I_p - U)^{b-(p+1)/2}, \quad 0 < U < I_p, \quad (1.3)$$

where $a > (p-1)/2$, $b > (p-1)/2$, and $\beta_p(a, b)$ is the multivariate beta function given by

$$\beta_p(a, b) = \frac{\Gamma_p(a) \Gamma_p(b)}{\Gamma_p(a+b)}, \quad (1.4)$$

where

$$\Gamma_p(a) = \pi^{p(p-1)/4} \prod_{j=1}^p \Gamma\left(a - \frac{j-1}{2}\right), \quad \operatorname{Re}(a) > \frac{p-1}{2}. \quad (1.5)$$

DEFINITION 1.2. A $p \times p$ random symmetric positive definite matrix V is said to have a matrix-variate beta type II distribution with parameters (a, b) , denoted as $V \sim B_p^{II}(a, b)$, if its pdf is given by

$$(\beta_p(a, b))^{-1} \det(V)^{a-(p+1)/2} \det(I_p + V)^{-(a+b)}, \quad V > 0, \quad (1.6)$$

where $a > (p-1)/2$, $b > (p-1)/2$, and $\beta_p(a, b)$ is the multivariate beta function.

As in the univariate case, the density (1.6) can be obtained from (1.3) by transforming $U = (I_p + V)^{-1}V$, together with the Jacobian $J(U \rightarrow V) = \det(I_p + V)^{-(p+1)}$.

The matrix-variate beta type I and type II distributions have been studied by many authors, e.g., see [7, 9, 10, 13, 14].

In this paper, a new matrix-variate beta distribution has been defined. We call it “Matrix-variate beta type III” distribution, which is then derived by using matrix transformation. Several properties of this distribution and its relationship with matrix-variate beta type I and type II distributions have also been studied.

2. Density function. First, we define the matrix-variate beta distribution of type III.

DEFINITION 2.1. A $p \times p$ random symmetric positive definite matrix W is said to have a matrix-variate beta type III distribution with parameters (a, b) , denoted as $W \sim B_p^{III}(a, b)$, if its pdf is given by

$$2^{pb} (\beta_p(a, b))^{-1} \det(W)^{b-(p+1)/2} \times \det(I_p - W)^{a-(p+1)/2} \det(I_p + W)^{-(a+b)}, \quad 0 < W < I_p, \quad (2.1)$$

where $a > (p-1)/2$, $b > (p-1)/2$, and $\beta_p(a, b)$ is the multivariate beta function.

For $p = 1$, the beta type III density is given by

$$2^b (\beta(a, b))^{-1} w^{b-1} (1-w)^{a-1} (1+w)^{-(a+b)}, \quad 0 < w < 1, \quad (2.2)$$

and in this case we write $w \sim B^{III}(a, b)$.

By means of a bilinear transformation of the random matrix U , the matrix-variate beta type III distribution is generated as in the following theorem.

THEOREM 2.1. Let $U \sim B_p^I(a, b)$. Define $W = (I_p + U)^{-1}(I_p - U)$. Then $W \sim B_p^{III}(a, b)$.

PROOF. Making the transformation $W = (I_p + U)^{-1}(I_p - U)$ with the Jacobian [12] in the pdf (1.3) of U , $J(U \rightarrow W) = 2^{p(p+1)/2} (I_p + W)^{-(p+1)}$, we get the desired result. $\square$

From the density of beta type III matrix it is apparent that

$$\int_{0 < W < I_p} \det(W)^{b-(p+1)/2} \det(I_p - W)^{a-(p+1)/2} \times \det(I_p + W)^{-(a+b)} dW = 2^{-pb} \beta_p(a, b). \quad (2.3)$$

The cumulative distribution function (cdf) of W is obtained as

$$P(W < \Omega) = 2^{pb} (\beta_p(a, b))^{-1} \int_{0 < W < \Omega} \det(W)^{b-(p+1)/2} \det(I_p - W)^{a-(p+1)/2} \\ \times \det(I_p + W)^{-(a+b)} dW. \quad (2.4)$$

For evaluating the above integral, we use the following results involving zonal polynomials [2, 5, 8]:

$$\det(I_p - W)^{a-(p+1)/2} = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(-a + (p+1)/2)_{\kappa}}{k!} C_{\kappa}(W), \quad (2.5)$$

$$\det(I_p + W)^{-(a+b)} = \sum_{t=0}^{\infty} \sum_{\tau} \frac{(a+b)_{\tau}}{t!} (-1)^t C_{\tau}(W), \quad (2.6)$$

$$C_{\kappa}(W) C_{\tau}(W) = \sum_{\delta} g_{\kappa, \tau}^{\delta} C_{\delta}(W), \quad (2.7)$$

$$\int_{0 < W < I_p} \det(W)^{a-(p+1)/2} \det(I_p - W)^{b-(p+1)/2} C_{\kappa}(RW) dW \\ = \beta_p(a, b) \frac{(a)_{\kappa}}{(a+b)_{\kappa}} C_{\kappa}(R), \quad \operatorname{Re}(a) > \frac{p-1}{2}, \operatorname{Re}(b) > \frac{p-1}{2}, \quad (2.8)$$

where $\kappa = (k_1, \dots, k_p)$, $k_1 \geq \dots \geq k_p \geq 0$, $k_1 + \dots + k_p = k$, $\tau = (t_1, \dots, t_p)$, $t_1 \geq \dots \geq t_p \geq 0$, $t_1 + \dots + t_p = t$, $\delta = (d_1, \dots, d_p)$, $d_1 \geq \dots \geq d_p \geq 0$, $d_1 + \dots + d_p = d = k + t$, and $g_{\kappa, \tau}^{\delta}$ is the coefficient of $C_{\delta}(W)$ in $C_{\kappa}(W) C_{\tau}(W)$. Substituting (2.5) and (2.6) in (2.4) and subsequently using (2.7), we obtain

$$P(W < \Omega) = 2^{pb} (\beta_p(a, b))^{-1} \sum_{k=0}^{\infty} \sum_{t=0}^{\infty} \sum_{\kappa} \sum_{\tau} \frac{(-a + (p+1)/2)_{\kappa}}{k!} \\ \times \frac{(a+b)_{\tau}}{t!} (-1)^t \sum_{\delta} g_{\kappa, \tau}^{\delta} \int_{0 < W < \Omega} \det(W)^{b-(p+1)/2} C_{\delta}(W) dW. \quad (2.9)$$

Now substituting $X = \Omega^{-1/2} W \Omega^{-1/2}$ with the Jacobian $J(W \rightarrow X) = \det(\Omega)^{(p+1)/2}$ in the above integral, we get

$$P(W < \Omega) = 2^{pb} (\beta_p(a, b))^{-1} \det(\Omega)^b \sum_{k=0}^{\infty} \sum_{t=0}^{\infty} \sum_{\kappa} \sum_{\tau} \frac{(-a + (p+1)/2)_{\kappa}}{k!} \\ \times \frac{(a+b)_{\tau}}{t!} (-1)^t \sum_{\delta} g_{\kappa, \tau}^{\delta} \int_{0 < X < I_p} \det(X)^{b-(p+1)/2} C_{\delta}(\Omega X) dX \\ = 2^{pb} \frac{\Gamma_p(a+b) \Gamma_p[(p+1)/2]}{\Gamma_p(a) \Gamma_p[b + (p+1)/2]} \det(\Omega)^b \sum_{k=0}^{\infty} \sum_{t=0}^{\infty} \sum_{\kappa} \sum_{\tau} \frac{(-a + (p+1)/2)_{\kappa}}{k!} \\ \times \frac{(a+b)_{\tau}}{t!} (-1)^t \sum_{\delta} g_{\kappa, \tau}^{\delta} \frac{(b)_{\delta}}{(b + (p+1)/2)_{\delta}} C_{\delta}(\Omega), \quad (2.10)$$

where the last step has been obtained by using (2.8).

The Laplace transform of the density of W is

$$L(Z) = 2^{pb} (\beta_p(a, b))^{-1} \int_{0 < W < I_p} \text{etr}(-ZW) \det(W)^{b-(p+1)/2} \\ \times \det(I_p - W)^{a-(p+1)/2} \det(I_p + W)^{-(a+b)} dW, \quad (2.11)$$

where $Z(p \times p) = ((1 + \delta_{ij})z_{ij}/2)$. Now, using the expansions

$$\det(I_p + W)^{-(a+b)} = 2^{-p(a+b)} \det\left(I_p - \frac{I_p - W}{2}\right)^{-(a+b)} \\ = 2^{-p(a+b)} \sum_{t=0}^{\infty} \sum_{\tau} \frac{(a+b)_{\tau}}{t!} \left(\frac{1}{2}\right)^t C_{\tau}(I_p - W), \quad (2.12) \\ \text{etr}(-ZW) = \text{etr}(-Z) \text{etr}\{Z(I_p - W)\} = \text{etr}(-Z) \sum_{k=0}^{\infty} \sum_{\kappa} \frac{1}{k!} C_{\kappa}(Z(I_p - W)),$$

in (2.11), we obtain

$$L(Z) = 2^{-pa} (\beta_p(a, b))^{-1} \text{etr}(-Z) \sum_{k=0}^{\infty} \sum_{t=0}^{\infty} \sum_{\kappa} \sum_{\tau} \frac{1}{k!} \frac{(a+b)_{\tau}}{2^t t!} \Phi(Z), \quad (2.13)$$

where

$$\Phi(Z) = \int_{0 < W < I_p} \det(W)^{b-(p+1)/2} \det(I_p - W)^{a-(p+1)/2} C_{\kappa}(Z(I_p - W)) C_{\tau}(I_p - W) dW \\ = \int_{0 < X < I_p} \det(I_p - X)^{b-(p+1)/2} \det(X)^{a-(p+1)/2} C_{\kappa}(ZX) C_{\tau}(X) dX. \quad (2.14)$$

Since $\Phi(Z) = \Phi(H'ZH)$, $H \in O(p)$, integrating out H in $\Phi(H'ZH)$ using [5, equation 23], we have

$$\Phi(Z) = \frac{C_{\kappa}(Z)}{C_{\kappa}(I_p)} \int_{0 < X < I_p} \det(I_p - X)^{b-(p+1)/2} \det(X)^{a-(p+1)/2} C_{\kappa}(X) C_{\tau}(X) dX. \quad (2.15)$$

Now using (2.7), and integrating X using (2.8), we obtain

$$\Phi(Z) = \frac{C_{\kappa}(Z)}{C_{\kappa}(I_p)} \sum_{\delta} g_{\kappa, \tau}^{\delta} \frac{(a)_{\delta}}{(a+b)_{\delta}} \frac{\Gamma_p(a) \Gamma_p(b)}{\Gamma_p(a+b)} C_{\delta}(I_p). \quad (2.16)$$

Substituting (2.16) in (2.13), we finally obtain

$$L(Z) = 2^{-pa} \text{etr}(-Z) \sum_{k=0}^{\infty} \sum_{t=0}^{\infty} \sum_{\kappa} \sum_{\tau} \frac{1}{k!} \frac{(a+b)_{\tau}}{2^t t!} \frac{C_{\kappa}(Z)}{C_{\kappa}(I_p)} \sum_{\delta} g_{\kappa, \tau}^{\delta} \frac{(a)_{\delta}}{(a+b)_{\delta}} C_{\delta}(I_p). \quad (2.17)$$

3. Properties. In this section, we study some properties of the random matrix distributed as matrix-variate beta type III.

THEOREM 3.1. Let $W \sim B_p^{\text{III}}(a, b)$ and $A (p \times p)$ be a constant nonsingular matrix. Then the density of $X = AWA'$ is

$$2^{pb} (\beta_p(a, b))^{-1} \det(AA')^{(p+1)/2} \det(X)^{b-(p+1)/2} \times \det(AA' - X)^{a-(p+1)/2} \det(AA' + X)^{-(a+b)}, \quad 0 < X < AA'. \quad (3.1)$$

PROOF. Making the transformation $X = AWA'$ with the Jacobian $J(W \rightarrow X) = \det(A)^{-(p+1)}$ in the pdf (2.1) of W , the density of X is obtained as

$$2^{pb} (\beta_p(a, b))^{-1} \det(AA')^{(p+1)/2} \det(X)^{b-(p+1)/2} \times \det(AA' - X)^{a-(p+1)/2} \det(AA' + X)^{-(a+b)}, \quad 0 < X < AA'. \quad (3.2)$$

which is the desired result. $\square$

We will write $X \sim B_p^{\text{III}}(a, b; AA')$. In the next theorem, it is shown that the matrix-variate beta distribution of type III is orthogonally invariant.

THEOREM 3.2. Let $W \sim B_p^{\text{III}}(a, b)$ and $H (p \times p)$ be an orthogonal matrix, whose elements are either constants or random variables distributed independently of W . Then, the distribution of W is invariant under the transformation $W \rightarrow HWH'$, and does not depend on H .

PROOF. First, let H be a constant orthogonal matrix. Then, from Theorem 3.1, $HWH' \sim B_p^{\text{III}}(a, b)$ since $HH' = I_p$. If, however, H is a random orthogonal matrix, then the conditional distribution of $HUH' \mid H \sim B_p^{\text{III}}(a, b)$. Since this distribution does not depend on H , $HWH' \sim B_p^{\text{III}}(a, b)$. $\square$

The relationship between beta type I, type II, and type III matrices is now exhibited. First, we derive the density of W^{-1} .

THEOREM 3.3. Let $W \sim B_p^{\text{III}}(a, b)$. Then the density of $Y = W^{-1}$ is

$$2^{pb} (\beta_p(a, b))^{-1} \det(Y - I_p)^{a-(p+1)/2} \det(I_p + Y)^{-(a+b)}, \quad Y > I_p. \quad (3.3)$$

PROOF. Making the transformation $Y = W^{-1}$ with the Jacobian $J(W \rightarrow Y) = \det(Y)^{-(p+1)}$, in the density of W the result follows. $\square$

The density derived above may be called the inverse beta type III. From Theorem 3.3, it is clear that if $W \sim B_p^{\text{III}}(a, b)$, then W^{-1} does not follow beta type III distribution.

THEOREM 3.4. (i) Let $U \sim B_p^{\text{I}}(a, b)$ and $W = (I_p + U)^{-1}(I_p - U)$, then $W \sim B_p^{\text{III}}(a, b)$.
(ii) Similarly, if $W \sim B_p^{\text{III}}(a, b)$ and $U = (I_p + W)^{-1}(I_p - W)$, then $U \sim B_p^{\text{I}}(a, b)$.

PROOF. Part (i) is proved in Theorem 2.1. Since $W = (I_p + U)^{-1}(I_p - U)$, the Jacobian of transformation is $J(W \rightarrow U) = 2^{p(p+1)/2} \det(I_p + U)^{-(p+1)}$. Now, making the substitution in the density of W given by (2.1) the result (ii) follows. $\square$

THEOREM 3.5. (i) Let $V \sim B_p^{\text{II}}(a, b)$ and $W = (I_p + 2V)^{-1}$, then $W \sim B_p^{\text{III}}(a, b)$.
(ii) Similarly, if $W \sim B_p^{\text{III}}(a, b)$ and $V = (I_p - W)W^{-1}/2$, then $V \sim B_p^{\text{II}}(a, b)$.

The marginal distributions of W is given in the following.

THEOREM 3.6. Let $W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}$, $W_{11} (q \times q)$, and $W_{22 \cdot 1} = W_{22} - W_{21} W_{11}^{-1} W_{12}$. If $W \sim B_p^{\text{III}}(a, b)$, then $W_{22 \cdot 1} \sim B_{p-q}^{\text{III}}(a, b - q/2)$.

PROOF. From the partition of W , we have

$$\det(W) = \det(W_{11}) \det(W_{22 \cdot 1}), \quad (3.4)$$

$$\det(I_p - W) = \det(I_q - W_{11}) \det(I_{p-q} - W_{22 \cdot 1} - W_{21} W_{11}^{-1} (I_q - W_{11})^{-1} W_{12}), \quad (3.5)$$

$$\det(I_p + W) = \det(I_q + W_{11}) \det(I_{p-q} + W_{22 \cdot 1} + W_{21} W_{11}^{-1} (I_q + W_{11})^{-1} W_{12}). \quad (3.6)$$

Now making the transformation $W_{11} = W_{11}$, $X = W_{21} W_{11}^{-1/2}$, and $W_{22 \cdot 1} = W_{22} - W_{21} W_{11}^{-1} W_{12} = W_{22} - XX'$ with Jacobian $J(W_{11}, W_{22}, W_{21} \rightarrow W_{11}, W_{22 \cdot 1}, X) = \det(W_{11})^{(p-q)/2}$ and substituting (3.4), (3.5), and (3.6) in the density of W , we get the joint density of W_{11} , $W_{22 \cdot 1}$, and X as follows:

$$\begin{aligned} & 2^{pb} (\beta_p(a, b))^{-1} \det(W_{11})^{b-(q+1)/2} \det(I_q - W_{11})^{a-(p+1)/2} \det(I_q + W_{11})^{-(a+b)} \\ & \times \det(W_{22 \cdot 1})^{b-(p+1)/2} \det(I_{p-q} - W_{22 \cdot 1})^{a-(p+1)/2} \det(I_{p-q} + W_{22 \cdot 1})^{-(a+b)} \\ & \times \det(I_{p-q} - (I_{p-q} - W_{22 \cdot 1})^{-1} X (I_q - W_{11})^{-1} X')^{a-(p+1)/2} \\ & \times \det(I_{p-q} + (I_{p-q} + W_{22 \cdot 1})^{-1} X (I_q + W_{11})^{-1} X')^{-(a+b)}. \end{aligned} \quad (3.7)$$

Substituting $Y = (I_{p-q} - W_{22 \cdot 1})^{-1/2} X (I_q - W_{11})^{-1/2}$ with Jacobian $J(X \rightarrow Y) = \det(I_{p-q} - W_{22 \cdot 1})^{q/2} \det(I_q - W_{11})^{(p-q)/2}$ and integrating Y , we get the joint density of W_{11} and $W_{22 \cdot 1}$ as

$$\begin{aligned} & 2^{pb} (\beta_p(a, b))^{-1} \det(W_{11})^{b-(q+1)/2} \det(I_q - W_{11})^{a-(q+1)/2} \det(I_q + W_{11})^{-(a+b)} \\ & \times \det(W_{22 \cdot 1})^{b-(p+1)/2} \det(I_{p-q} - W_{22 \cdot 1})^{a-(p-q+1)/2} \det(I_{p-q} + W_{22 \cdot 1})^{-(a+b)} g(A, B), \end{aligned} \quad (3.8)$$

where

$$\begin{aligned} g(A, B) &= \int_{I_{p-q} - YY' > 0} \det(I_{p-q} - YY')^{a-(p+1)/2} \det(I_{p-q} + AYBY')^{-(a+b)} dY \\ &= \int_{0 < Z < I_{p-q}} \int_{YY' = Z} \det(I_{p-q} - YY')^{a-(p+1)/2} {}_1F_0^{(q)}(a+b; -Y'AYB) dY dZ, \quad \text{for } p-q \leq q, \\ &= \int_{0 < Z < I_q} \int_{Y'Y = Z} \det(I_q - Y'Y)^{a-(p+1)/2} {}_1F_0^{(p-q)}(a+b; -AYBY') dY dZ, \quad \text{for } p-q > q, \end{aligned} \quad (3.9)$$

with $A = (I_{p-q} - W_{22 \cdot 1})^{1/2} (I_{p-q} + W_{22 \cdot 1})^{-1} (I_{p-q} - W_{22 \cdot 1})^{1/2}$ and $B = (I_q - W_{11})^{1/2} (I_q + W_{11})^{-1} (I_q - W_{11})^{1/2}$. Since $g(A, B) = g(A, H'BH)$, $H \in O(q)$, by integrating H in $g(A, H'BH)$, we obtain

$$\begin{aligned}
g(A, B) &= \int_{0 < Z < I_{p-q}} \int_{YY' = Z} \det(I_{p-q} - YY')^{a-(p+1)/2} {}_1F_0^{(q)}(a+b; -AYY', B) dY dZ \\
&= \frac{\pi^{q(p-q)/2}}{\Gamma_{p-q}(q/2)} \int_{0 < Z < I_{p-q}} \det(Z)^{(q-p+q-1)/2} \det(I_{p-q} - Z)^{a-(p+1)/2} {}_1F_0^{(q)}(a+b; -AZ, B) dZ \\
&= \frac{\pi^{q(p-q)/2}}{\Gamma_{p-q}(q/2)} \cdot \frac{\Gamma_{p-q}(q/2) \Gamma_{p-q}(a-q/2)}{\Gamma_{p-q}(a)} {}_2F_1^{(q)}\left(\frac{q}{2}, a+b; a; -A, B\right) \\
&= \frac{\pi^{q(p-q)/2} \Gamma_{p-q}(a-q/2)}{\Gamma_{p-q}(a)} {}_2F_1^{(q)}\left(\frac{q}{2}, a+b; a; -A, B\right). \tag{3.10}
\end{aligned}$$

Substituting $g(A, B)$ in (3.8), we get the joint density of W_{11} and $W_{22 \cdot 1}$ as

$$\begin{aligned}
&2^{qb} (\beta_q(a, b))^{-1} \det(W_{11})^{b-(q+1)/2} \det(I_q - W_{11})^{a-(q+1)/2} \\
&\quad \times \det(I_q + W_{11})^{-(a+b)} 2^{(p-q)(b-q/2)} \left(\beta_{p-q}\left(a, b - \frac{q}{2}\right)\right)^{-1} \det(W_{22 \cdot 1})^{b-(p+1)/2} \\
&\quad \times \det(I_{p-q} - W_{22 \cdot 1})^{a-(p-q+1)/2} \det(I_{p-q} + W_{22 \cdot 1})^{-(a+b)} 2^{(p-q)q/2} \\
&\quad \times {}_2F_1^{(q)}\left(\frac{q}{2}, a+b; a; -A, B\right). \tag{3.11}
\end{aligned}$$

Clearly W_{11} and $W_{22 \cdot 1}$ are not independent. Integrating W_{11} , using

$$\begin{aligned}
&2^{qb} \int_{0 < W_{11} < I_q} \det(W_{11})^{b-(q+1)/2} \det(I_q - W_{11})^{a-(q+1)/2} \det(I_q + W_{11})^{-(a+b)} \\
&\quad \times {}_2F_1^{(q)}\left(\frac{q}{2}, a+b; a; -A, B\right) dW_{11} \\
&= \int_{0 < B < I_q} \det(B)^{a-(q+1)/2} \det(I_q - B)^{b-(q+1)/2} {}_2F_1\left(\frac{q}{2}, a+b; a; -A, B\right) dB \\
&= \beta_q(a, b) {}_3F_2^{(q)}\left(a, \frac{q}{2}, a+b; a, a+b; -A\right) \\
&= \beta_q(a, b) \det(I_{p-q} + A)^{-q/2} \\
&= 2^{-(p-q)q/2} \beta_q(a, b) \det(I_{p-q} + W_{22 \cdot 1})^{q/2}, \tag{3.12}
\end{aligned}$$

we get the density of $W_{22 \cdot 1}$. For $q < p - q$, using (3.9) and following similar steps we get the same result. $\square$

Alternately, Theorem 3.6 can be proved using the relationship between type II and type III beta matrices. Let $V \sim B_p^{\text{II}}(a, b)$ and $V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}$, $V_{11} (q \times q)$ and $V_{11 \cdot 2} = V_{11} - V_{12} V_{22}^{-1} V_{21}$. It is well known that $V_{11 \cdot 2}$ and V_{22} are distributed independently (see [4]), $V_{11 \cdot 2} \sim B_q^{\text{II}}(a - (p - q)/2, b)$ and $V_{22} \sim B_{p-q}^{\text{II}}(a, b - q/2)$. According to Theorem 3.5(i), if $V \sim B_p^{\text{II}}(a, b)$, then $W = (I_p + 2V)^{-1} \sim B_p^{\text{III}}(a, b)$. Furthermore,

$$W^{-1} = \begin{pmatrix} W^{11} & W^{12} \\ W^{21} & W^{22} \end{pmatrix} = \begin{pmatrix} I_q + 2V_{11} & 2V_{12} \\ 2V_{21} & I_{p-q} + 2V_{22} \end{pmatrix}. \tag{3.13}$$

That is, $W^{22} ((p - q) \times (p - q)) = W_{22 \cdot 1}^{-1} = I_{p-q} + 2V_{22}$ and $W_{22 \cdot 1} = (I_q + 2V_{22})^{-1}$.

Now, since $V_{22} \sim B_{p-q}^{II}(a, b - q/2)$, then $W_{22 \cdot 1} = (I_{p-q} + 2V_{22})^{-1} \sim B_{p-q}^{III}(a, b - q/2)$.

The distribution of $(AW^{-1}A')^{-1}$ where $A(q \times p)$ is a constant matrix of rank $q (\leq p)$, is now derived.

THEOREM 3.7. *Let $A(q \times p)$ be a constant matrix of rank $q (\leq p)$. If $W \sim B_p^{III}(a, b)$, then $(AW^{-1}A')^{-1} \sim B_q^{III}(a, b - (p - q)/2; (AA')^{-1})$.*

PROOF. We write $A = M(I_q \ 0)G$, where $M(q \times q)$ is nonsingular and $G(p \times p)$ is orthogonal. Now,

$$\begin{aligned} (AW^{-1}A')^{-1} &= (M(I_q \ 0)GW^{-1}G'(I_q \ 0)'M')^{-1} \\ &= (M')^{-1} \left[(I_q \ 0)Y^{-1} \begin{pmatrix} I_q \\ 0 \end{pmatrix} \right]^{-1} M^{-1} \\ &= (M')^{-1} (Y^{11})^{-1} M^{-1}, \end{aligned} \quad (3.14)$$

where $Y = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} = GWG' \sim B_p^{III}(a, b)$, $Y_{11}(q \times q)$ and $Y^{11} = (Y_{11} - Y_{12}Y_{22}^{-1}Y_{21})^{-1} = Y_{11 \cdot 2}^{-1}$. From Theorem 3.6, $Y_{11 \cdot 2} \sim B_q^{III}(a, b - (p - q)/2)$ and from Theorem 3.1, $(M')^{-1}Y_{22 \cdot 1}M^{-1} \sim B_q^{III}(a, b - (p - q)/2; (MM')^{-1})$. The proof is now completed by observing that $MM' = AA'$. $\square$

COROLLARY 3.8. *Let $W \sim B_p^{III}(a, b)$ and $\mathbf{a} \in \Re^p$, $\mathbf{a} \neq \mathbf{0}$, then $\mathbf{a}'\mathbf{a}/\mathbf{a}'W^{-1}\mathbf{a} \sim B^{III}(a, b - (p - 1)/2)$.*

In Corollary 3.8, the distribution of $\mathbf{a}'\mathbf{a}/\mathbf{a}'W^{-1}\mathbf{a}$ does not depend on $\mathbf{a}$. Thus, if $\mathbf{y}(p \times 1)$ is a random vector, independent of W , and $P(\mathbf{y} \neq \mathbf{0}) = 1$, then it follows that $\mathbf{y}'\mathbf{y}/\mathbf{y}'W^{-1}\mathbf{y} \sim B^{III}(a, b - (p - 1)/2)$.

THEOREM 3.9. *Let $W \sim B_p^{III}(a, b)$, then*

$$\begin{aligned} \text{(i)} \quad E \left[\frac{\det(W)^h}{\det(I_p + W)^h} \right] &= \frac{\Gamma_p(a + b)\Gamma_p(b + h)}{2^{ph}\Gamma_p(b)(a + b + h)}, \quad \text{Re}(h) > -b + \frac{p-1}{2}, \\ \text{(ii)} \quad E[\det(W)^h] &= \frac{2^{-pa}\Gamma_p(a + b)\Gamma_p(b + h)}{\Gamma_p(b)\Gamma_p(a + b + h)} {}_2F_1(a, a + b; a + b + h, \tfrac{1}{2}I_p), \quad \text{Re}(h) > -b + \frac{p-1}{2}, \\ \text{(iii)} \quad E[\det(I_p - W)^h] &= \frac{2^{-pa}\Gamma_p(a + b)\Gamma_p(a + h)}{\Gamma_p(a)\Gamma_p(a + b + h)} {}_2F_1(a + h, a + b; a + b + h, \tfrac{1}{2}I_p), \quad \text{Re}(h) > -a + \frac{p-1}{2}, \end{aligned}$$

where ${}_2F_1$ is the hypergeometric function of matrix argument.

PROOF. (i) From the density of W , we have

$$\begin{aligned} E \left[\frac{\det(W)^h}{\det(I_p + W)^h} \right] &= 2^{pb}(\beta_p(a, b))^{-1} \int_{0 < W < I_p} \det(W)^{b+h-(p+1)/2} \det(I_p - W)^{a-(p+1)/2} \\ &\quad \times \det(I_p + W)^{-(a+b+h)} dW \\ &= 2^{pb}(\beta_p(a, b))^{-1} \frac{\beta_p(a, b + h)}{2^{p(b+h)}}, \quad \text{Re}(h) > -b + \frac{p-1}{2}. \end{aligned} \quad (3.15)$$

Simplifying this last expression using (1.4), we get the desired result.

(ii) From the density of W , we have

$$E[\det(W)^h] = 2^{pb} (\beta_p(a, b))^{-1} \times \int_{0 < W < I_p} \det(W)^{b+h-(p+1)/2} \det(I_p - W)^{a-(p+1)/2} \det(I_p + W)^{-(a+b)} dW. \quad (3.16)$$

Writing $\det(I_p + W)^{-(a+b)}$ in series involving zonal polynomials using (2.12), we obtain

$$\begin{aligned} E[\det(W)^h] &= 2^{-pa} (\beta_p(a, b))^{-1} \sum_{t=0}^{\infty} \sum_{\tau} \frac{(a+b)_{\tau}}{2^t t!} \\ &\times \int_{0 < W < I_p} \det(W)^{b+h-(p+1)/2} \det(I_p - W)^{a-(p+1)/2} C_{\tau}(I_p - W) dW \\ &= 2^{-pa} \frac{\beta_p(a, b+h)}{\beta_p(a, b)} \sum_{t=0}^{\infty} \sum_{\tau} \frac{(a+b)_{\tau}}{s^t t!} \cdot \frac{a_{\tau}}{(a+b+h)_{\tau}} C_{\tau}(I_p), \quad \operatorname{Re}(h) > -b + \frac{p-1}{2}, \end{aligned} \quad (3.17)$$

where the integral has been evaluated using (2.8). Finally, simplifying the expression using results on hypergeometric functions [2, 5], we get the desired result.

Similarly $E[\det(I_p - W)^h]$ can be derived. $\square$

From the density of W , we have

$$E[C_{\kappa}(W)] = 2^{pb} (\beta_p(a, b))^{-1} \int_{0 < W < I_p} C_{\kappa}(W) \det(W)^{b-(p+1)/2} \times \det(I_p - W)^{a-(p+1)/2} \det(I_p + W)^{-(a+b)} dW. \quad (3.18)$$

Writing $\det(I_p + W)^{-(a+b)}$ in series involving zonal polynomials using (2.6), we obtain

$$\begin{aligned} E[C_{\kappa}(W)] &= 2^{pb} (\beta_p(a, b))^{-1} \sum_{t=0}^{\infty} \sum_{\tau} \frac{(a+b)_{\tau}}{t!} (-1)^t \\ &\times \int_{0 < W < I_p} C_{\kappa}(W) C_{\tau}(W) \det(W)^{b-(p+1)/2} \det(I_p - W)^{a-(p+1)/2} dW \\ &= 2^{pb} (\beta_p(a, b))^{-1} \sum_{t=0}^{\infty} \sum_{\tau} \frac{(a+b)_{\tau}}{t!} (-1)^t \sum_{\delta} g_{\kappa, \tau}^{\delta} \\ &\times \int_{0 < W < I_p} C_{\delta}(W) \det(W)^{b-(p+1)/2} \det(I_p - W)^{a-(p+1)/2} dW \\ &= 2^{pb} \sum_{t=0}^{\infty} \sum_{\tau} \frac{(a+b)_{\tau}}{t!} (-1)^t \sum_{\delta} g_{\kappa, \tau}^{\delta} \frac{(b)_{\delta}}{(a+b)_{\delta}} C_{\delta}(I_p), \end{aligned} \quad (3.19)$$

where the last two steps have been obtained using (2.7) and (2.8).

THEOREM 3.10. (i) Let $U \sim B_p^I(a, b)$ and $W \sim B_p^{III}(c, a + b)$ be independent. Then $Y = W^{1/2}UW^{1/2}$, $0 < Y < I_p$ has the density

$$2^{p(a+b)} (\beta_p(a, b+c))^{-1} \det(Y)^{a-(p+1)/2} \det(I_p - Y)^{b+c-(p+1)/2} \\ \times \det(I_p + Y)^{-(a+b+c)} {}_2F_1(b, a+b+c; b+c; -(I_p + Y)^{-1}(I_p - Y)), \quad 0 < Y < I_p. \quad (3.20)$$

Further, pdf of $X = (I_p + Y)^{-1}(I_p - Y) = (I_p + W^{1/2}UW^{1/2})^{-1}(I_p - W^{1/2}UW^{1/2})$ is

$$2^{pb} (\beta_p(a, b+c))^{-1} \det(X)^{b+c-(p+1)/2} \det(I_p - X)^{a-(p+1)/2} \\ \times {}_2F_1(b, a+b+c; b+c; -X), \quad 0 < X < I_p. \quad (3.21)$$

(ii) Let $U \sim B_p^I(a, b)$ and $V \sim B_p^I(a+b, c)$ be independent. Then $Y = (I_p + V^{1/2}UV^{1/2})^{-1} \times (I_p - V^{1/2}UV^{1/2}) \sim B_p^{III}(a, b+c)$.

PROOF. The joint density of U and V is

$$2^{p(a+b)} (\beta_p(a, b)\beta_p(c, a+b))^{-1} \det(U)^{a-(p+1)/2} \det(I_p - U)^{b-(p+1)/2} \\ \times \det(W)^{a+b-(p+1)/2} \det(I_p - W)^{c-(p+1)/2} \\ \times \det(I_p + W)^{-(a+b+c)}, \quad 0 < U < I_p, \quad 0 < W < I_p. \quad (3.22)$$

Making the transformation $Y = W^{1/2}UW^{1/2}$ with Jacobian $J(U \rightarrow Y) = \det(W)^{-(p+1)/2}$ in above, we get the joint density of Y and W as

$$2^{p(a+b)} (\beta_p(a, b)\beta_p(c, a+b))^{-1} \det(Y)^{a-(p+1)/2} \det(W - Y)^{b-(p+1)/2} \\ \times \det(I_p - W)^{c-(p+1)/2} \det(I_p + W)^{-(a+b+c)}, \quad 0 < Y < W < I_p. \quad (3.23)$$

Substituting $Z = (I_p - Y)^{-1/2}(W - Y)(I_p - Y)^{-1/2}$ with the Jacobian $J(W \rightarrow Z) = \det(I_p - Y)^{(p+1)/2}$ and integrating Z , we get the density of Y as

$$2^{p(a+b)} (\beta_p(a, b)\beta_p(c, a+b))^{-1} \det(Y)^{a-(p+1)/2} \\ \times \det(I_p - Y)^{b+c-(p+1)/2} \det(I_p + Y)^{-(a+b+c)} \\ \times \int_{0 < Z < I_p} \det(Z)^{b-(p+1)/2} \det(I_p - Z)^{c-(p+1)/2} \\ \times \det(I_p + (I_p + Y)^{-1}(I_p - Y)Z)^{-(a+b+c)} dZ. \quad (3.24)$$

Integration of Z using [5, equation 48] completes the proof.

(ii) From [6], we have $V^{1/2}UV^{1/2} \sim B_p^I(a, b+c)$ and from Theorem 2.1, we have $(I_p + V^{1/2}UV^{1/2})^{-1}(I_p - V^{1/2}UV^{1/2}) \sim B_p^{III}(a, b+c)$. $\square$

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