NEW CHARACTERIZATIONS OF SOME L^p -SPACES

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ABSTRACT. For a complete measure space (X, Σ, μ) , we give conditions which force $L^p(X, \mu)$, for $1 \leq p < \infty$, to be isometrically isomorphic to $\ell^p(\Gamma)$ for some index set Γ which depends only on (X, μ) . Also, we give some new characterizations which yield the inclusion $L^p(X, \mu) \subset L^q(X, \mu)$ for $0 < p < q$.

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1. Introduction. Suppose X is a nonempty set, Σ is σ -algebra of subsets of X , μ a positive measure on Σ . For each positive number p , let $L^p(X, \mu)$ denote the space of all real valued Σ -measurable functions f on X such that $\int_X |f|^p d\mu < \infty$, and $L^\infty(X, \mu)$ denote the space of all essentially bounded, real valued Σ -measurable functions on X . In [2, 3, 5] some characterizations of the positive measure μ on (X, Σ) for which $L^p(X, \mu) \subseteq L^q(X, \mu)$, $0 < p < q$, are given. The purpose of this note is to give some new characterizations of such measure μ which yield the inclusion $L^p(X, \mu) \subseteq L^q(X, \mu)$ for $0 < p < q$. Our proofs are more transparent, direct, and work even if the measure μ is not σ -finite. Further we show that in a situation when $L^p(X, \mu) \subseteq L^q(X, \mu)$ for some pair p, q with $0 < p < q$, then $L^p(X, \mu)$, for $1 \leq p < \infty$, is isometrically isomorphic to $\ell^p(\Gamma)$ for some index set Γ which depends only on the measure space (X, Σ, μ) .

2. Preliminaries. Throughout the following (X, Σ, μ) is a positive measure space. We assume that the measure μ is complete. For the sake of simplicity, we write $L^p(\mu)$ for $L^p(X, \mu)$ and $L^\infty(\mu)$ for $L^\infty(X, \mu)$. A set $A \in \Sigma$ is called an *atom* if $\mu(A) > 0$ and for every $E \subset A$ with $E \in \Sigma$, either $\mu(E) = 0$ or $\mu(E) = \mu(A)$. A measurable subset E with $\mu(E) > 0$ is *nonatomic* if it does not contain any atom. We say that two atoms A_1 and A_2 are *distinct* if $\mu(A_1 \cap A_2) = 0$. We say that two atoms A_1 and A_2 are *indistinguishable* if $\mu(A_1 \cap A_2) = \mu(A_1) = \mu(A_2)$. A measurable space (X, Σ, μ) is said to be *atomic* if every measurable set of positive measure contains an atom. For more information on measurable spaces and related topics we refer to [1, 2, 4]. We collect some interesting and useful properties of atomic and nonatomic sets in the following proposition.

PROPOSITION 2.1. *Let (X, Σ, μ) be a complete measure space.*

(a) *If $\{A_n\}$ is a sequence of distinct atoms, then there exists a sequence $\{B_n\}$ of disjoint atoms such that for each n , $B_n \subseteq A_n$ and $\bigcup_{n=1}^\infty A_n = \bigcup_{n=1}^\infty B_n$.*

(b) *If $\{A_n\}$ is a sequence of distinct atoms, and A is an atom contained in $\bigcup A_n$, then there exists a unique m such that A is indistinguishable from A_m .*

(c) If A is a nonatomic set of positive measure, then there exists a sequence $\{E_n\}$ of disjoint measurable subsets of A of positive measure such that $\mu(E_n) \rightarrow 0$ as $n \rightarrow \infty$.

(d) If $f \in L^p(\mu)$ and A is an atom in Σ , then f is constant almost everywhere (a.e.) on A .

PROOF. (a) Let $B_1 = A_1$ and $B_n = A_n \setminus \bigcup_{k=1}^{n-1} A_k$. Obviously B_i 's are disjoint and $\bigcup A_n = \bigcup B_n$. Also $\mu(B_n) = \mu(A_n \setminus \bigcup_{k=1}^{n-1} A_k)$ is either zero or is equal to $\mu(A_n)$. If $\mu(B_n) = 0$, then $\mu(A_n) = \mu(A_n \cap (\bigcup_{k=1}^{n-1} A_k)) \leq \sum_{k=1}^{n-1} \mu(A_n \cap A_k)$. Since A_k 's are distinct atoms, this implies $\mu(A_n) = 0$ which is absurd. Hence $\mu(B_n) = \mu(A_n)$.

(b) Suppose A is contained in $\bigcup A_n$. From part (a) of the proposition, there exists a sequence $\{B_n\}$ of disjoint atoms such that $B_n \subseteq A_n$ for each n and $\bigcup A_n = \bigcup B_n$. Obviously

$$\mu(A) = \sum_{n=1}^{\infty} \mu(A \cap B_n). \quad (2.1)$$

Clearly $\mu(A \cap B_n)$ is either zero or $\mu(A)$ for each n . Hence by (2.1), there exists a unique m such that $\mu(A \cap B_m) = \mu(A)$. Since A and B_m are indistinguishable, $B_m \subset A_m$, it follows that A and A_m are indistinguishable.

(c) Suppose A is a nonatomic set of positive measure and $\mu(A) = \delta$. There exists a measurable subset E_1 of A such that $0 < \mu(E_1) < \delta/2$. Since $A \setminus E_1$ is nonatomic, there exists a measurable subset E_2 of $A \setminus E_1$ such that $0 < \mu(E_2) < \delta/4$. Having chosen $E_1, E_2, \dots, E_{n-1}$, choose a measurable subset E_n of $A \setminus (E_1 \cup E_2 \cup \dots \cup E_{n-1})$ such that $\mu(E_n) < \delta/2^n$. Obviously E_n 's are disjoint and $\mu(E_n) \rightarrow 0$ as $n \rightarrow \infty$.

(d) Since A is an atom, it is enough to show that if f is integrable then f is constant a.e. on A . Choose a real number c such that $c\mu(A) = \int_A f(x) d\mu$. Let $B = \{x \in A \mid f(x) \neq c\}$. We claim $\mu(B) = 0$. Obviously $B = \{x \in A \mid f(x) < c\} \cup \{x \in A \mid f(x) > c\}$. First, we show that $\mu(\{x \in A \mid f(x) > c\}) = 0$. We can use a similar argument to show that $\mu(\{x \in A \mid f(x) < c\}) = 0$. We note that $\{x \in A \mid f(x) > c\} = \bigcup_{i=1}^{\infty} B_i \cup B_0$, where $B_i = \{x \in A \mid c + 1/(1+i) \leq f(x) < c + (1/i)\}$ and $B_0 = \{x \in A \mid f(x) \geq c + 1\}$. Obviously all B_i 's are disjoint. Since A is an atom, at most one of the B_i 's can have a positive measure. If B_k is of positive measure for some k , $0 \leq k < \infty$, then $c\mu(A) = \int_A f(x) d\mu(x) = \int_{B_k} f(x) d\mu(x) \geq (c + 1/(k+1))\mu(A)$. This is absurd. Therefore, $\mu(B_i) = 0$ for all $i \geq 0$. Hence $\{x \in A \mid f(x) > c\}$ is of measure zero. This completes the proof. $\square$

The following lemmas are quite useful in the proof of the main result.

LEMMA 2.2. Let (X, Σ, μ) be a complete measure space.

(a) If $\{B_n\}$ is a sequence of measurable sets of positive measure and $\mu(B_n) \rightarrow 0$ as $n \rightarrow \infty$, then there exists a sequence $\{C_n\}$ of disjoint measurable sets of positive measure such that $\mu(C_n) \rightarrow 0$ as $n \rightarrow \infty$.

(b) If $\{E_n\}$ is a sequence of disjoint measurable sets of positive measure such that $\mu(E_n) \rightarrow 0$ as $n \rightarrow \infty$, then for any positive number $m > 1$ there exists a subsequence $\{E_{n_i}\}$ of $\{E_n\}$ and an increasing sequence $\{k_i\}$ of positive integers such that $\mu(E_{n_i}) \in ((1/k_i)^m, (1/k_i)^{m-1}]$.

PROOF. (a) Without loss of generality, we may assume that $\mu(B_n) < 1$ for each n . If for some positive integer k , B_k is nonatomic, by using an argument similar to

that of Proposition 2.1(c), we can construct a sequence C_n of disjoint measurable sets of positive measure such that $\mu(C_n) \rightarrow 0$ as $n \rightarrow \infty$. Suppose that B_k is atomic for each positive integer k , let A_1 be an atom contained in B_1 . Since $\mu(B_n) \rightarrow 0$ as $n \rightarrow \infty$, $\mu(A_1 \cap B_k)$ can be positive only for finitely many $k > 1$. Let n_1 be the smallest positive integer such that $\mu(A_1 \cap B_{n_1}) = 0$. Now choose an atom A_2 contained in B_{n_1} . Obviously A_2 is indistinguishable from A_1 . Also, $\mu(A_2 \cap B_k)$ can be positive for at most finitely many k greater than n_1 . Let n_2 be the smallest positive integer greater than n_1 such that $\mu(A_2 \cap B_{n_2}) = 0$. Now choose an atom A_3 contained in B_{n_2} . Clearly A_3 is indistinguishable from A_1 and A_2 . Continuing in this fashion, we get a sequence $\{A_k\}$ of atoms which are indistinguishable and $A_k \subseteq B_{n_{k-1}}$ for each $k \geq 2$. By Proposition 2.1(a), we may choose a sequence $\{E_k\}$ of disjoint atoms such that $E_k \subseteq A_k$. Clearly, $0 < \mu(E_k) = \mu(A_k) \leq \mu(B_{n_{k-1}})$. This completes the proof of part (a).

(b) Let $\{E_n\}$ be a sequence of measurable sets of positive measure such that $\mu(E_n) \rightarrow 0$ as $n \rightarrow \infty$. Without loss of generality, we may assume that $\{\mu(E_n)\}$ is a strictly decreasing sequence. Let $m > 1$. Let $k_0 > 2$ be a positive integer such that $1/2 < (k/(k+1))^{m-1}$ for all $k \geq k_0$. Clearly $(1/(\ell+1))^m, 1/(\ell+1)^{m-1}] \cap ((1/\ell)^m, (1/\ell)^{m-1}]$ is nonempty for each $\ell \geq k_0$. Since $\mu(E_n)$ is decreasing to zero, the set $\{\mu(E_n) \mid n \geq 1\}$ must have a nonempty intersection with an interval $((1/k)^m, (1/k)^{m-1}]$ for some $k \geq k_0$. Let k_1 be the smallest positive integer greater than k_0 such that $\{\mu(E_n) \mid n \geq 1\} \cap ((1/k_1)^m, (1/k_1)^{m-1}] \neq \emptyset$. Let n_1 be the smallest positive integer such that $\mu(E_{n_1}) \in ((1/k_1)^m, (1/k_1)^{m-1}]$. Next choose the smallest integer k_2 greater than k_1 such that $\{\mu(E_n) \mid n > n_1\} \cap ((1/k_2)^m, (1/k_2)^{m-1}] \neq \emptyset$. Let n_2 be the smallest integer greater than n_1 such that $\mu(E_{n_2}) \in ((1/k_2)^m, (1/k_2)^{m-1}]$. Continuing inductively in this way, we can choose strictly increasing sequences of positive integers $\{k_i\}$ and $\{n_i\}$ such that $\mu(E_{n_i}) \in ((1/k_i)^m, (1/k_i)^{m-1}]$. This completes the proof of part (b). $\square$

LEMMA 2.3. *If $L^p(\mu) \subseteq L^q(\mu)$ for $0 < p < q$, then there does not exist a disjoint sequence $\{E_n\}$ of measurable sets of positive measure such that $\mu(E_n) \rightarrow 0$ as $n \rightarrow \infty$.*

PROOF. Suppose there exists a disjoint sequence $\{E_n\}$ of measurable sets of positive measure such that $\mu(E_n) \rightarrow 0$ as $n \rightarrow \infty$. Let

$$m = 3 - \frac{3p}{p-q} = -\frac{3q}{p-q}. \quad (2.2)$$

Clearly $m > 1$. By Lemma 2.2(b), there exists a subsequence $\{E_{n_i}\}$ of $\{E_n\}$ and a strictly increasing sequence of positive integers $\{k_i\}$ such that $\mu(E_{n_i}) \in ((1/k_i)^m, (1/k_i)^{m-1}]$. Define a function f from X into real numbers by $f(x) = (1/k_i)^{3/(p-q)}$ if $x \in E_{n_i}$ and $f(x) = 0$ for all $x \notin \cup_{i=1}^{\infty} E_{n_i}$. Then

$$\begin{aligned} \int_X |f(x)|^p d\mu &= \sum_{i=1}^{\infty} \int_{E_{n_i}} |f(x)|^p d\mu = \sum_{i=1}^{\infty} \left(\frac{1}{k_i}\right)^{3p/(p-q)} \mu(E_{n_i}) \\ &\leq \sum_{i=1}^{\infty} \left(\frac{1}{k_i}\right)^{3p/(p-q)} \left(\frac{1}{k_i}\right)^{m-1} = \sum_{i=1}^{\infty} \left(\frac{1}{k_i}\right)^2 < \infty. \end{aligned} \quad (2.3)$$

On the other hand,

$$\begin{aligned} \int_X |f(x)|^q d\mu &= \sum_{i=1}^{\infty} \int_{E_{n_i}} |f(x)|^q d\mu = \sum_{i=1}^{\infty} \left(\frac{1}{k_i}\right)^{3q/(p-q)} \mu(E_{n_i}) \\ &\geq \sum_{i=1}^{\infty} \left(\frac{1}{k_i}\right)^{3q/(p-q)} \left(\frac{1}{k_i}\right)^m = \infty. \end{aligned} \quad (2.4)$$

Thus $f \in L^p(\mu)$ but $f \notin L^q(\mu)$. This completes the proof of the lemma. $\square$

3. Main results. For the sake of clarity, we first start with a definition. For any nonempty set Γ , and $p > 0$, we define $\ell^p(\Gamma)$ to be the set of all extended real valued functions f on Γ such that f is nonzero only on a countable subset of Γ and $\sum_{\alpha} |f(\alpha)|^p < \infty$.

When $p \geq 1$, $\ell^p(\Gamma)$ becomes a Banach space under the norm defined by $\|f\|_{\ell^p(\Gamma)} = (\sum_{\alpha} |f(\alpha)|^p)^{1/p}$. Now, we are ready to state the main result.

THEOREM 3.1. *Let (X, Σ, μ) be a complete measure space. The following six conditions are equivalent:*

- (1) $L^p(\mu) \subset L^q(\mu)$ for some pair of real numbers p and q with $0 < p < q$.
- (2) $L^p(\mu) \subset L^\infty(\mu)$ for some $p > 0$.
- (3) $L^p(\mu) \subset L^\infty(\mu)$ for all positive numbers p .
- (4) $L^p(\mu) \subset L^q(\mu)$ for all p and q with $0 < p < q$.
- (5) There is no sequence $\{B_n\}$ in Σ such that $\mu(B_n) > 0$ for each n and $\mu(B_n) \rightarrow 0$ as $n \rightarrow \infty$.
- (6) (X, Σ, μ) is atomic with $\inf_{A \in \Pi} \mu(A) > 0$, where Π is the set of all atoms in Σ .

Moreover, these statements imply that: for each positive number $p \geq 1$, $L^p(\mu)$ is isometrically isomorphic to $\ell^p(\Gamma)$ for some index set Γ which depends only on (X, Σ, μ) .

PROOF. Since the implication (4) $\Rightarrow$ (1) is obvious, in order to prove the equivalence of the statements (1) through (6), it is enough to prove the following implications: (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3), (3) $\Rightarrow$ (4), (4) $\Rightarrow$ (5), (5) $\Rightarrow$ (6), and (6) $\Rightarrow$ (2).

(1) $\Rightarrow$ (2): suppose that $L^p \subset L^q$ for some pair p, q with $0 < p < q$. We claim $L^p \subset L^\infty$. Suppose there is an f in L^p which is not essentially bounded. Then there exists a strictly increasing sequence $\{n_k\}$ of positive integers such that for each $k \geq 1$, the set $E_k = \{x \in X \mid n_k \leq |f(x)| < n_k + 1\}$ is of a positive measure. Obviously E_k 's are disjoint. Since $\mu(E_k)n_k^p \leq \int_{E_k} |f|^p d\mu \leq \int_X |f|^p d\mu$, it follows $\mu(E_k) \rightarrow 0$. This is a contradiction in view of Lemma 2.2.

(2) $\Rightarrow$ (3): suppose that $L^p(\mu) \subset L^\infty(\mu)$ for some $p > 0$. Let r be any positive real number. We show $L^r(\mu) \subset L^\infty(\mu)$. let $f \in L^r(\mu)$. If $A = \{x : |f(x)| > 1\}$ is of measure zero, then obviously $f \in L^\infty(\mu)$. Suppose that A is a positive measure. Let $g = X_A f$, where X_A is the characteristic function of the set A . Clearly, $g \in L^r(\mu)$ and $|g| \geq 1$ a.e. Since $|g|^{r/p} \in L^p$, $|g|^{r/p} \in L^\infty$. Let $M = \text{ess sup } |g|^{r/p}$. Let $\epsilon > 0$. Choose $\delta > 0$ such that $(M + \delta)^{p/r} - M^{p/r} < \epsilon$. Since $\{x : |g(x)| > M^{p/r} + \epsilon\} \subseteq \{x : |g(x)| > (M + \delta)^{p/r}\}$, and $\mu(\{x : |g(x)|^{r/p} > M + \delta\}) = 0$, it follows that $\text{ess sup } |g| \leq M^{p/r}$.

(3) $\Rightarrow$ (4): suppose that $L^p \subset L^\infty$ for all $p > 0$. Let $g \in L^p$. Write $A = \{x : |g(x)| > 1\}$. If A is a nonatomic set of positive measure, by Proposition 2.1(c), A contains a disjoint

sequence $\{E_n\}$ of measurable subsets of A of positive measure such that $\mu(E_n) \rightarrow 0$ as $n \rightarrow \infty$. As is noted in the proof of Lemma 2.3, we can construct a function f in L^p which is not in L^∞ . Hence A contains an atom. Since the measure of A is finite, in view of Proposition 2.1(a), A cannot contain infinitely many atoms. Therefore, A can be written as a finite disjoint union of atoms. Suppose that $A = \cup_{i=1}^n \theta_i$, where θ_i 's are disjoint atoms. By Proposition 2.1(d), g is constant on each θ_i . Let g_{θ_i} be the value of g on θ_i . Then for any $q > p$,

$$\begin{aligned} \int_X |g|^q du &= \int_{X-A} |g|^q du + \int_A |g|^q du \\ &\leq \int_{X-A} |g|^p du + \sum_{i=1}^n |g_{\theta_i}|^q \mu(\theta_i) \\ &\leq \int_X |g|^p du + \sum_{i=1}^n |g_{\theta_i}|^q \mu(\theta_i) < \infty. \end{aligned} \quad (3.1)$$

Hence $L^p \subset L^q$ for $q > p$.

(4) $\Rightarrow$ (5): this follows from Lemmas 2.2(a) and 2.3.

(5) $\Rightarrow$ (6): Proposition 2.1(c) implies that the space (X, Σ, μ) is atomic. Since atoms are of positive measure, obviously statement (5) implies that $\inf_{A \in \pi} \mu(A) > 0$.

(6) $\Rightarrow$ (2): Suppose (X, Σ, μ) is atomic with $\inf_{A \in \pi} \mu(A) > 0$. Let $p > 0$ and $g \in L^p(\mu)$. Suppose $B = \{x | g(x) > 1\}$. If $\mu(B) = 0$, then clearly $g \in L^\infty$. Suppose $\mu(B) > 0$. Obviously $\mu(B)$ is finite. Since $\inf_{A \in \pi} \mu(A) > 0$, B cannot contain infinitely many atoms. Therefore, B can be written as finite disjoint union of atoms. Since g is constant on each atom, it follows that $g \in L^\infty$.

Finally, we show that for $p \geq 1$, one of the statements (1) through (6) (and hence all of them) imply statement (7). Let (X, Σ, μ) be a measure space such that $L^p(\mu) \subseteq L^q(\mu)$ for some $1 \leq p < q$. Let $\{\theta_i\}_{i \in \Gamma}$ be the collection of all atoms in X where Γ is some index set. Let $f \in L^p(\mu)$ be an arbitrary nonzero element of f . By Proposition 2.1(d) f is constant almost everywhere on any atom. We denote the value of f on an atom θ lies in the support of f by f_θ . Since the support of f is σ -finite, and by statement (5) of the theorem any measurable set of finite measure is disjoint union of finitely many atoms, the support of f can be written as countable union of atoms. Let $\{\theta_n(f)\}$ be the set of all atoms that forms the support of f . We define $F : L^p(\mu) \rightarrow \ell^p(\Gamma)$ by

$$F(f)(\gamma) = \begin{cases} f_{\theta_n} (\mu(\theta_n))^{1/p}, & \text{if } \theta_\gamma = \theta_n(f) \text{ for some } n, \\ 0, & \text{if } \theta_\gamma \notin \{\theta_n(f)\} \end{cases} \quad (3.2)$$

for any nonzero f in $L^p(\mu)$. The function F is well defined since any two functions that are equal in $L^p(\mu)$ are equal almost everywhere and thus share the same support. It is straightforward to verify that F is a one-to-one linear operator from $L^p(\mu)$ into $\ell^p(\Gamma)$. Let $h \in \ell^p(\Gamma)$. Since h is nonzero only on a countable subset Γ_h of Γ , define f on X as follows:

$$f(x) = \begin{cases} \frac{h(\gamma)}{(\mu(\theta_\gamma))^{1/p}}, & \text{if } x \in \theta_\gamma, \gamma \in \Gamma_h, \\ 0, & \text{if } x \notin \bigcup_{\gamma \in \Gamma_h} \theta_\gamma. \end{cases} \quad (3.3)$$

Obviously, $f \in L^p(\mu)$ and $F(f) = h$. Thus F is an isomorphism from $L^p(\mu)$ onto $\ell^p(\Gamma)$. Further for any $f \in L^p(\mu)$,

$$\begin{aligned} \|F(f)\|_{\ell^p(\Gamma)}^p &= \sum_i |f_{\theta_i}(\mu(\theta_i))^{(1/p)}|^p = \sum_i |f_{\theta_i}|^p \mu(\theta_i) \\ &= \sum_i \int_{\theta_i} |f(x)|^p d\mu = \int_X |f(x)|^p d\mu = \|f\|^p, \end{aligned} \quad (3.4)$$

where the sum runs over $i \in \Gamma$ such that θ_i is in the support of f .

Therefore F is an isometry. This completes the proof of the theorem. $\square$

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