APPROXIMATING FIXED POINTS OF λ -FIRMLY NONEXPANSIVE MAPPINGS IN BANACH SPACES

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ABSTRACT. We study the convergence of the Ishikawa iteration methods to fixed points for the result of Smarzewski (1991). Our theorems also improve recent theorems due to Sharma and Sahu (1996).

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1. Introduction. Let E be a real Banach space and let C be a nonempty closed convex subset of E . Then a mapping T of C into itself is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A mapping T of C into itself is called *λ -firmly nonexpansive* if there exists $\lambda \in (0, 1)$ such that

$$\|Tx - Ty\| \leq \|(1 - \lambda)(x - y) + \lambda(Tx - Ty)\| \quad \forall x, y \in C. \quad (1.1)$$

It is clear that every λ -firmly nonexpansive mapping is nonexpansive. For a mapping T of C into itself, we consider the following iteration scheme: $x_1 \in C$,

$$x_{n+1} = \alpha_n T[\beta_n Tx_n + (1 - \beta_n)x_n] + (1 - \alpha_n)x_n \quad \forall n \geq 1, \quad (1.2)$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1]$. Such an iteration scheme was introduced by Ishikawa [5]; see also Mann iteration scheme (corresponding to the choice $\beta_n = 0$ for all $n \in \mathbb{N}$) [6]. Now let C be a nonempty convex subset of a Banach space E , and let T, S be mappings of C into itself. Then, for an $x_1 \in C$, we consider the iterates $\{x_n\}$ defined by

$$\begin{aligned} x_{n+1} &= \alpha_n Ty_n + (1 - \alpha_n)Sx_n, \\ y_n &= \beta_n Tx_n + (1 - \beta_n)x_n \quad \forall n \geq 1, \end{aligned} \quad (1.3)$$

where α_n and β_n satisfy $0 < a \leq \alpha_n$, $\beta_n \leq b < 1$. If $S = I$, the identity mapping, the iterates (1.3) are reduced to the above special case due to Ishikawa [5]. In 1991, Smarzewski [10] proved the following result: let E be a uniformly convex Banach space and let $C = \bigcup_{i=1}^n C_i$ be a union of nonempty bounded closed convex subsets C_i of E and suppose $T : C \rightarrow C$ is λ -firmly nonexpansive for some $\lambda \in (0, 1)$. Then T has a fixed point in C . The result above is no longer true if T is merely nonexpansive, even in one-dimensional space; see [10]. Recently, Sharma and Sahu [9] studied the

convergence of the Mann and Ishikawa iteration methods to fixed points for the result of Smarzewski [10].

In this paper, we first show that the iterates $\{x_n\}$ and $\{y_n\}$ defined by (1.3) converge weakly to the same common fixed point of T and S when E is a uniformly convex Banach space with Opial's condition or Fréchet differentiable norm. Next, we show that the iterates $\{x_n\}$ defined by (1.2) converge weakly to a fixed point of T when E is a uniformly convex Banach space with Opial's condition. Finally, we show that if E is uniform convex and if the ranges of T are contained in a compact subset of C , the iterates $\{x_n\}$ defined by (1.2) converge strongly to a fixed point of T . This paper also improves recent theorems due to Sharma and Sahu [9] using ideas of Takahashi-Kim [12].

2. Preliminaries. Throughout this paper, we denote by E and E^* a real Banach space and the dual space of E , respectively. The value of $x^* \in E^*$ at $x \in E$ is denoted by $\langle x, x^* \rangle$. Let C be a nonempty closed convex subset of E and let T be a mapping from C into itself. Then we denote by $F(T)$ the set of all fixed points of T , i.e., $F(T) = \{x \in C : Tx = x\}$. We also denote by $\mathbb{N}$ the set of all natural numbers and by $\mathbb{R}$ and $\mathbb{R}^+$ the sets of all real numbers and all nonnegative real numbers, respectively. $\overline{\text{co}}A$ means the closure of the convex hull of A . A Banach space E is called *uniformly convex* if for each $\epsilon > 0$ there is a $\delta > 0$ such that for $x, y \in E$ with $\|x\|, \|y\| \leq 1$ and $\|x - y\| \geq \epsilon$, $\|x + y\| \leq 2(1 - \delta)$ holds. When $\{x_n\}$ is a sequence in E , then $x_n \rightarrow x$ (resp., $x_n \rightharpoonup x$, $x_n \xrightarrow{*} x$) denote strong (resp., weak, *weak**) convergence of the sequence $\{x_n\}$ to x . A Banach space E is said to satisfy *Opial's condition* [7] if for any sequence $\{x_n\}$ in E , $x_n \rightharpoonup x$ implies that

$$\limsup_{n \rightarrow \infty} \|x_n - x\| < \limsup_{n \rightarrow \infty} \|x_n - y\| \quad \forall y \in E \text{ with } y \neq x. \quad (2.1)$$

If $I - T$ is demiclosed at zero [1], i.e., for any sequence $\{x_n\}$ in C , the conditions $x_n \rightarrow x$ weakly and $x_n - Tx_n \rightarrow 0$ strongly imply $x - Tx = 0$. With each $x \in E$, we associate the set

$$J_\phi(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\| \|x^*\| \text{ and } \|x^*\| = \phi(\|x\|)\}, \quad (2.2)$$

where $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a continuous and strictly increasing function with $\phi(0) = 0$ and $\phi(\infty) = \infty$. Then $J_\phi : E \rightarrow 2^{E^*}$ is said to be the duality mapping. Suppose that J_ϕ is single-valued. Then J_ϕ is said to be weakly sequentially continuous if for each $\{x_n\} \in E$ with $x_n \rightharpoonup x$, then $J_\phi(x_n) \xrightarrow{*} J_\phi(x)$. For abbreviation, we set $J := J_\phi$. In all our proofs we assume, without loss of generality, that J is normalized. We know that if E admits a weakly sequentially continuous duality mapping, then E satisfies Opial's condition; see [4]. Let $S(E) = \{x \in E : \|x\| = 1\}$. Then the norm of E is said to be *Gâteaux differentiable* (and E is said to be *smooth*) if

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \quad (2.3)$$

exists for each x and y in $S(E)$. It is also said to be *Fréchet differentiable* if, for each $x \in S(E)$, the limit (2.3) is attained uniformly in $y \in S(E)$. All Hilbert spaces and

l^p ($1 < p < \infty$) satisfy Opial's condition, while L^p with $1 < p \neq 2 < \infty$ do not. It is well known that if E is smooth, then the duality mapping J is single-valued and strong-weak* continuous; for more details, see [2] or [11].

3. Convergence theorems. We first begin with the following.

LEMMA 3.1 (see [8]). *Let E be a uniformly convex Banach space, $0 < b \leq t_n \leq c < 1$ for all $n \geq 1$, and $a \geq 0$. Suppose that $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ are sequences of E such that $\limsup_{n \rightarrow \infty} \|x_n\| \leq a$, $\limsup_{n \rightarrow \infty} \|y_n\| \leq a$, and $\lim_{n \rightarrow \infty} \|t_n x_n + (1 - t_n) y_n\| = a$. Then $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.*

Using Lemma 3.1, we have the following.

LEMMA 3.2. *Let $C = \bigcup_{i=1}^n C_i$ be a union of nonempty closed convex subsets C_i of a uniformly convex Banach space E and let $T, S : C \rightarrow C$ be λ -firmly nonexpansive for some $\lambda \in (0, 1)$ and $tT(sTx + (1 - s)x) + (1 - t)Sx \in C$ for all $x \in C$ and $s, t \in (0, 1)$. Then $F(T) \cap F(S)$ is nonempty if and only if the iterates $\{x_n\}$ defined by (1.3) is bounded, $\{x_n - Tx_n\}$ and $\{x_n - Sx_n\}$ converge strongly to zero as $n \rightarrow \infty$.*

PROOF. Let w be a common fixed point of T and S . Since T and S are λ -firmly nonexpansive for some $\lambda \in (0, 1)$, it is easy to check that $\|x_{n+1} - w\| \leq \|x_n - w\|$ for all $n \geq 1$. So, $\{x_n\}$ is bounded and $\lim_{n \rightarrow \infty} \|x_n - w\|$ exists. Put $c = \lim_{n \rightarrow \infty} \|x_n - w\|$. Since T is λ -firmly nonexpansive for some $\lambda \in (0, 1)$, we obtain

$$\begin{aligned} \|Ty_n - w\| &\leq \|(1 - \lambda)(y_n - w) + \lambda(Ty_n - w)\| \\ &\leq (1 - \lambda)\|y_n - w\| + \lambda\|Ty_n - w\|, \end{aligned} \quad (3.1)$$

and thus $\|Ty_n - w\| \leq \|y_n - w\|$. Taking $\limsup_{n \rightarrow \infty}$ in both sides, we obtain

$$\limsup_{n \rightarrow \infty} \|Ty_n - w\| \leq \limsup_{n \rightarrow \infty} \|y_n - w\| \leq \limsup_{n \rightarrow \infty} \|x_n - w\| = c. \quad (3.2)$$

Furthermore, since

$$\lim_{n \rightarrow \infty} \|\alpha_n(Ty_n - w) + (1 - \alpha_n)(Sx_n - w)\| = \lim_{n \rightarrow \infty} \|x_{n+1} - w\| = c, \quad (3.3)$$

by Lemma 3.1, we have $\lim_{n \rightarrow \infty} \|Ty_n - Sx_n\| = 0$. Since

$$\begin{aligned} \|x_{n+1} - w\| &\leq \alpha_n \|Ty_n - w\| + (1 - \alpha_n) \|x_n - w\| \\ &\leq \alpha_n \|y_n - w\| + (1 - \alpha_n) \|x_n - w\|, \end{aligned} \quad (3.4)$$

we have

$$\frac{\|x_{n+1} - w\| - \|x_n - w\|}{\alpha_n} \leq \|y_n - w\| - \|x_n - w\|. \quad (3.5)$$

Since $\{\alpha_n\}$ is assumed to be bounded away from zero, we obtain

$$c \leq \liminf_{n \rightarrow \infty} \|y_n - w\|. \quad (3.6)$$

Since $\|y_n - w\| \leq \|x_n - w\|$ for all $n \geq 1$, we have

$$c = \lim_{n \rightarrow \infty} \|y_n - w\| = \lim_{n \rightarrow \infty} \|\beta_n(Tx_n - w) + (1 - \beta_n)(x_n - w)\|. \quad (3.7)$$

By Lemma 3.1, we have $\lim_{n \rightarrow \infty} \|Tx_n - x_n\| = 0$. Since

$$\begin{aligned} \|x_n - Sx_n\| &\leq \|x_n - Tx_n\| + \|Tx_n - Ty_n\| + \|Ty_n - Sx_n\| \\ &\leq (1 + \beta_n)\|x_n - Tx_n\| + \|Ty_n - Sx_n\|, \end{aligned} \quad (3.8)$$

we have $x_n - Sx_n \rightarrow 0$ as $n \rightarrow \infty$.

Conversely, suppose that $\{x_n\}$ is bounded, $\{x_n - Tx_n\}$ and $\{x_n - Sx_n\}$ converge strongly to zero as $n \rightarrow \infty$. Then we can consider a real-valued function g on C given by

$$g(v) = \limsup_{n \rightarrow \infty} \|x_n - v\| \quad \text{for each } v \in C. \quad (3.9)$$

By [11], we know that $g : C \rightarrow \mathbb{R}$ is continuous and convex. Further, if $\|v_n\| \rightarrow \infty$, then $g(v_n) \rightarrow \infty$. So, we have an element $v_0 \in C$ such that $g(v_0) = r = \min_{v \in C} g(v)$. Set $M = \{v_0 \in C : r = g(v_0)\}$. Then M is bounded, closed, and convex. Further, M is invariant under T . In fact, let $z \in M$. Then, for some $\lambda \in (0, 1)$, we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|Tx_n - Tz\| &\leq \limsup_{n \rightarrow \infty} \|(1 - \lambda)(x_n - z) + \lambda(Tx_n - Tz)\| \\ &\leq (1 - \lambda) \limsup_{n \rightarrow \infty} \|x_n - z\| + \lambda \limsup_{n \rightarrow \infty} \|Tx_n - Tz\| \end{aligned} \quad (3.10)$$

and thus

$$\limsup_{n \rightarrow \infty} \|x_n - Tz\| = \limsup_{n \rightarrow \infty} \|Tx_n - Tz\| \leq \limsup_{n \rightarrow \infty} \|x_n - z\|. \quad (3.11)$$

Hence $Tz \in M$. Similarly, M is invariant under S . Since E is uniformly convex and hence M consists of one point, we have a common fixed point of T and S in M ; see [13]. $\square$

REMARK 3.3. In Lemma 3.2, if $F(T) \cap F(S) \neq \emptyset$, then we furthermore see that $\{y_n - Ty_n\}$ and $\{y_n - Sy_n\}$ converge strongly to zero as $n \rightarrow \infty$.

We first consider the following weak convergence of λ -firmly nonexpansive mappings in a Banach space.

THEOREM 3.4. *Let E be a uniformly convex Banach space satisfying Opial's condition and let $C = \bigcup_{i=1}^n C_i$ be a union of nonempty closed convex subsets C_i of E and let $T, S : C \rightarrow C$ be λ -firmly nonexpansive for some $\lambda \in (0, 1)$ with a common fixed point and $tT(sTx + (1 - s)x) + (1 - t)Sx \in C$ for all $x \in C$ and $s, t \in (0, 1)$. Then the iterates $\{x_n\}$ and $\{y_n\}$ defined by (1.3) converge weakly to a common fixed point of T and S . Further, the two w -limits of $\{x_n\}$ and $\{y_n\}$ coincide.*

PROOF. Let z be a common fixed point of T and S . Then, as in the proof of Lemma 3.2, we have $\lim_{n \rightarrow \infty} \|x_n - z\|$ exists. Let z_1 and z_2 be two weak subsequential limits of the sequence $\{x_n\}$. We claim that the conditions $x_{n_i} \rightharpoonup z_1$ and $x_{n_j} \rightharpoonup z_2$ imply $z_1 = z_2 \in F(T) \cap F(S)$. We first show that $z_1, z_2 \in F(T)$. In fact, if $Tz_1 \neq z_1$, then, by Opial's condition, we have $\limsup_{i \rightarrow \infty} \|x_{n_i} - z_1\| < \limsup_{i \rightarrow \infty} \|x_{n_i} - Tz_1\|$. Since T is λ -firmly nonexpansive for some $\lambda \in (0, 1)$, we obtain

$$\begin{aligned} \limsup_{i \rightarrow \infty} \|Tx_{n_i} - Tz_1\| &\leq \limsup_{i \rightarrow \infty} \|(1 - \lambda)(x_{n_i} - z_1) + \lambda(Tx_{n_i} - Tz_1)\| \\ &\leq (1 - \lambda) \limsup_{i \rightarrow \infty} \|x_{n_i} - z_1\| + \lambda \limsup_{i \rightarrow \infty} \|Tx_{n_i} - Tz_1\|. \end{aligned} \quad (3.12)$$

By Lemma 3.2, we have

$$\limsup_{i \rightarrow \infty} \|x_{n_i} - Tz_1\| \leq \limsup_{i \rightarrow \infty} \|x_{n_i} - z_1\|. \quad (3.13)$$

This is a contradiction. Hence we have $Tz_1 = z_1$. Similarly, we have $z_2 \in F(T)$. Next, we show $z_1 = z_2$. If not, by Opial's condition,

$$\begin{aligned} \lim_{n \rightarrow \infty} \|x_n - z_1\| &= \lim_{i \rightarrow \infty} \|x_{n_i} - z_1\| < \lim_{i \rightarrow \infty} \|x_{n_i} - z_2\| \\ &= \lim_{n \rightarrow \infty} \|x_n - z_2\| = \lim_{j \rightarrow \infty} \|x_{n_j} - z_2\| \\ &< \lim_{j \rightarrow \infty} \|x_{n_j} - z_1\| = \lim_{n \rightarrow \infty} \|x_n - z_1\|. \end{aligned} \quad (3.14)$$

This is a contradiction. Hence we have $z_1 = z_2$. By using the same method as above, we have $z_1 = z_2 \in F(S)$. This implies that $\{x_n\}$ converges weakly to a common fixed point of T and S . As in the proof of Lemma 3.2, we have $\lim_{n \rightarrow \infty} \|y_n - z\|$ exists. Let $y_{n_i} \rightarrow w_1$ and $y_{n_j} \rightarrow w_2$. Then, by using the same method as above, we obtain $w_1 = w_2 \in F(T) \cap F(S)$. Further, since $\|x_n - y_n\| = \beta_n \|x_n - Tx_n\| \rightarrow 0$ as $n \rightarrow \infty$, we readily see that two w -limits of $\{x_n\}$ and $\{y_n\}$ coincide. $\square$

THEOREM 3.5. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm. Let $C = \bigcup_{i=1}^n C_i$ be a union of nonempty closed convex subsets C_i of E and let $T, S : C \rightarrow C$ be λ -firmly nonexpansive for some $\lambda \in (0, 1)$ with a common fixed point, and let $I - T, I - S$ be demiclosed at zero and $tT(sTx + (1-s)x) + (1-t)Sx \in C$ for all $x \in C$ and $s, t \in (0, 1)$. Then the iterates $\{x_n\}$ and $\{y_n\}$ defined by (1.3) converge weakly to a common fixed point of T and S . Further, the two w -limits of $\{x_n\}$ and $\{y_n\}$ coincide.*

PROOF. Since $F(T) \cap F(S)$ is nonempty, it follows from Lemma 3.2 that $\{x_n\}$ is bounded, $\{x_n - Tx_n\}$ and $\{x_n - Sx_n\}$ converge strongly to zero as $n \rightarrow \infty$. There exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and a point $z \in C$ such that $x_{n_i} \rightarrow z$. Since $I - T$ and $I - S$ are demiclosed at zero, we obtain $z \in F(T) \cap F(S)$. For $y, z \in F(T) \cap F(S)$, as in the proof of Lemma 2 [12], we have $\lim_{n \rightarrow \infty} \langle x_n, J(y - z) \rangle$ exists. To prove Theorem 3.5, assume $x_{n_i} \rightarrow z_1$ and $x_{n_j} \rightarrow z_2$. Then, for $y, z \in F(T) \cap F(S)$, we have

$$\begin{aligned} \langle z_1, J(y - z) \rangle &= \lim_{i \rightarrow \infty} \langle x_{n_i}, J(y - z) \rangle = \lim_{n \rightarrow \infty} \langle x_n, J(y - z) \rangle \\ &= \lim_{j \rightarrow \infty} \langle x_{n_j}, J(y - z) \rangle = \langle z_2, J(y - z) \rangle. \end{aligned} \quad (3.15)$$

Setting $y = z_1$ and $z = z_2$, we obtain $\langle z_1 - z_2, J(z_1 - z_2) \rangle = 0$ and hence $z_1 = z_2$. This implies that $\{x_n\}$ converges weakly to a common fixed point of T and S . By using the same method as above, $\{y_n\}$ converges weakly to a common fixed point of T and S . Further, since $x_n - y_n \rightarrow 0$ as $n \rightarrow \infty$, the remaining part of the proof is trivial. $\square$

THEOREM 3.6. *Let E be a uniformly convex Banach space satisfying Opial's condition, and let $C = \bigcup_{i=1}^n C_i$ be a union of nonempty bounded closed convex subsets C_i of E and let $T : C \rightarrow C$ be λ -firmly nonexpansive for some $\lambda \in (0, 1)$ and $tT(sTx + (1-s)x) + (1-t)x \in C$ for all $x \in C$ and $s, t \in (0, 1)$. Then for any initial data x_1 in C , the iterates $\{x_n\}$ defined by (1.2), where $\{\alpha_n\}$ and $\{\beta_n\}$ are chosen so that $\alpha_n \in [a, b]$ and*

$\beta_n \in [0, b]$ or $\alpha_n \in [a, 1]$ and $\beta_n \in [a, b]$ for some a, b with $0 < a \leq b < 1$, converge weakly to a fixed point of T .

PROOF. The existence of a fixed point follows from Smarzewski [10]. Let w be a fixed point of T . Then, as in the proof of Lemma 3.2, we have $\lim_{n \rightarrow \infty} \|x_n - w\|$ exists. Put $c = \lim_{n \rightarrow \infty} \|x_n - w\|$. Since T is λ -firmly nonexpansive for some $\lambda \in (0, 1)$, we obtain

$$\begin{aligned} \|Ty_n - w\| &\leq \|(1 - \lambda)(y_n - w) + \lambda(Ty_n - w)\| \\ &\leq (1 - \lambda)\|y_n - w\| + \lambda\|Ty_n - w\| \end{aligned} \quad (3.16)$$

and thus $\|Ty_n - w\| \leq \|y_n - w\|$. Taking $\limsup_{n \rightarrow \infty}$ in both sides, we obtain

$$\limsup_{n \rightarrow \infty} \|Ty_n - w\| \leq \limsup_{n \rightarrow \infty} \|x_n - w\| = c. \quad (3.17)$$

Further, we have

$$\lim_{n \rightarrow \infty} \|\alpha_n(Ty_n - w) + (1 - \alpha_n)(x_n - w)\| = \lim_{n \rightarrow \infty} \|x_{n+1} - w\| = c. \quad (3.18)$$

If $0 < a \leq \alpha_n \leq b < 1$ and $0 \leq \beta_n \leq b < 1$, by Lemma 3.1, we have $\lim_{n \rightarrow \infty} \|Ty_n - x_n\| = 0$. Since

$$\begin{aligned} \|Tx_n - x_n\| &\leq \|Tx_n - Ty_n\| + \|Ty_n - x_n\| \\ &\leq \|x_n - y_n\| + \|Ty_n - x_n\| \\ &\leq \beta_n \|Tx_n - x_n\| + \|Ty_n - x_n\|, \end{aligned} \quad (3.19)$$

we obtain

$$(1 - b)\|Tx_n - x_n\| \leq (1 - \beta_n)\|Tx_n - x_n\| \leq \|Ty_n - x_n\|. \quad (3.20)$$

Therefore $\|x_n - Tx_n\| \rightarrow 0$ as $n \rightarrow \infty$. On the other hand, we have, for all $n \geq 1$,

$$\begin{aligned} \|x_{n+1} - w\| &\leq \alpha_n \|Ty_n - w\| + (1 - \alpha_n) \|x_n - w\| \\ &\leq \alpha_n \|y_n - w\| + (1 - \alpha_n) \|x_n - w\| \end{aligned} \quad (3.21)$$

and hence

$$\frac{\|x_{n+1} - w\| - \|x_n - w\|}{\alpha_n} \leq \|y_n - w\| - \|x_n - w\|. \quad (3.22)$$

If $0 < a \leq \alpha_n \leq 1$ and $0 < a \leq \beta_n \leq b < 1$, we obtain

$$c \leq \liminf_{n \rightarrow \infty} \|y_n - w\|. \quad (3.23)$$

Since $\|y_n - w\| \leq \|x_n - w\|$ for all $n \geq 1$, we obtain

$$c = \lim_{n \rightarrow \infty} \|y_n - w\| = \lim_{n \rightarrow \infty} \|\beta_n(Tx_n - w) + (1 - \beta_n)(x_n - w)\|. \quad (3.24)$$

By Lemma 3.1, we have $\lim_{n \rightarrow \infty} \|Tx_n - x_n\| = 0$. As in the proof of Theorem 3.4, the result follows. $\square$

COROLLARY 3.7. Let E be a uniformly convex Banach space possessing a weakly sequentially continuous duality mapping and let $C = \bigcup_{i=1}^n C_i$ be a union of nonempty bounded closed convex subsets C_i of E and let $T : C \rightarrow C$ be a λ -firmly nonexpansive for some $\lambda \in (0, 1)$ and let $I - T$ be demiclosed at zero and $tTx + (1 - t)x \in C$ for all $x \in C$ and $t \in (0, 1)$. Let $\{\alpha_n\}$ be a real sequence satisfying $0 < a \leq \alpha_n \leq b < 1$ for all $n \in \mathbb{N}$.

Pick $x_1 \in C$ and define $x_{n+1} = \alpha_n T x_n + (1 - \alpha_n) x_n$ for all $n \in N$. Then $\{x_n\}$ converges weakly to a fixed point of T .

COROLLARY 3.8. Let E be a uniformly convex Banach space possessing a weakly sequentially continuous duality mapping and let $C = \bigcup_{i=1}^n C_i$ be a union of nonempty bounded closed convex subsets C_i of E and let $T : C \rightarrow C$ be λ -firmly nonexpansive for some $\lambda \in (0, 1)$ and let $I - T$ be demiclosed at zero and $tT(sTx + (1-s)x) + (1-t)x \in C$ for all $x \in C$ and $s, t \in (0, 1)$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequence real sequence satisfying $0 < a \leq \alpha_n \leq b < 1$ and $0 < c \leq \beta_n \leq d < 1$ for all $n \in N$. Pick $x_1 \in C$ and the iterates $\{x_n\}$ defined by (1.2). Then $\{x_n\}$ converges weakly to a fixed point of T .

Next, we consider a strong convergence of λ -firmly nonexpansive mapping in a Banach space.

THEOREM 3.9. Let E be a uniformly convex Banach space and let $C = \bigcup_{i=1}^n C_i$ be a union of nonempty bounded closed convex subsets C_i of E with $C_i \subseteq C_{i+1}$. Suppose that $T : C \rightarrow C$ is λ -firmly nonexpansive for some $\lambda \in (0, 1)$ such that $T(C)$ is contained in a compact subset of C . Then for any initial data x_1 in C , the iterates $\{x_n\}$ defined by (1.2), where $\{\alpha_n\}$ and $\{\beta_n\}$ are chosen so that $\alpha_n \in [a, b]$ and $\beta_n \in [0, b]$ or $\alpha_n \in [a, 1]$ and $\beta_n \in [a, b]$ for some a, b with $0 < a \leq b < 1$, converge strongly to a fixed point of T .

PROOF. Note that $\{x_n\}$ is well defined. The existence of a fixed point follows from Smarzewski [10]. By Mazur's theorem [3], $\overline{\text{co}}(\{x_1\} \cup T(C))$ is a compact subset of C containing $\{x_n\}$. There exist a subsequence $\{x_m\}$ of the sequence $\{x_n\}$ and a point $z \in C$ such that $x_m \rightarrow z$. As in the proof of Theorem 3.6, $\{x_n - Tx_n\}$ converges strongly to zero as $n \rightarrow \infty$. Since T is λ -firmly nonexpansive for some $\lambda \in (0, 1)$, we obtain

$$\begin{aligned} \|z - Tz\| &\leq \|z - x_m\| + \|x_m - Tx_m\| + \|Tx_m - Tz\| \\ &\leq 2\|z - x_m\| + \|x_m - Tx_m\| \rightarrow 0 \quad \text{as } m \rightarrow \infty. \end{aligned} \quad (3.25)$$

Hence $Tz = z$. As in the proof of Lemma 3.2, we have $\lim_{n \rightarrow \infty} \|x_n - z\|$ exists. Hence we have $\lim_{n \rightarrow \infty} \|x_n - z\| = 0$. $\square$

REMARK 3.10. In Theorem 3.9, if $T, S : C \rightarrow C$ are λ -firmly nonexpansive for some $\lambda \in (0, 1)$ such that $T(C)$ and $S(C)$ are contained in a compact subset of C and $F(T) \cap F(S) \neq \emptyset$, then the iterates $\{x_n\}$ and $\{y_n\}$ defined by (1.3) converge strongly to the same common fixed point of T and S .

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