DERIVATIONS OF CERTAIN OPERATOR ALGEBRAS

JIANKUI LI and HEMANT PENDHARKAR

(Received 20 March 1998 and in revised form 2 July 1999)

Abstract. Let $\mathcal{N}$ be a nest and let $\mathcal{A}$ be a subalgebra of $L(H)$ containing all rank one operators of alg $\mathcal{N}$. We give several conditions under which any derivation $\delta$ from $\mathcal{A}$ into $L(H)$ must be inner. The conditions include (1) $H_- \neq H$, (2) $0_+ \neq 0$, (3) there is a nontrivial projection in $\mathcal{N}$ which is in $\mathcal{A}$, and (4) $\delta$ is norm continuous. We also give some applications.

Keywords and phrases. Derivation, nest algebra, triangular operator algebra.

2000 Mathematics Subject Classification. Primary 47B47, 47L35.

1. Introduction. In this paper, we unify some results on derivations by considering derivations from an algebra $\mathcal{A}$ containing all rank one operators of a nest algebra into an $\mathcal{A}$-bimodule $\mathcal{B}$. Chernoff [1] proves that every derivation from $F(H)$ into $L(H)$ is inner. In [2], Christensen proves that every derivation from a nest algebra into itself or into $L(H)$ is inner. In [3], Christensen and Peligrad show that every derivation of a quasitriangular operator algebra into itself is inner. Knowles [7] generalizes the result of [2] and gets that any derivation from a nest algebra into an ideal $\mathcal{J}$ of $L(H)$ is inner. Let $\mathcal{N}$ be a nest of subspaces of a Hilbert space $H$, let $\mathcal{A}$ be a subalgebra of $L(H)$ containing all rank one operators of alg $\mathcal{N}$, and let $\delta$ be a derivation from $\mathcal{A}$ into $L(H)$. We prove that if one of the following conditions holds:

1. $H_- \neq H$,
2. $0_+ \neq 0$,
3. there exists a nontrivial $P \in \mathcal{N}$, such that $P \in \mathcal{A}$, then $\delta$ is inner.

We also prove that for any nest, if $\delta$ is a norm continuous derivation from $\mathcal{A}$ into $L(H)$, then $\delta$ is inner.

We discuss some applications of these results.

Let $H$ be a complex separable Hilbert space, $L(H)$ the algebra of all bounded linear operators on $H$, $K(H)$ the ideal of all compact operators in $L(H)$, $F(H)$ the subalgebra of all finite rank operators on $H$, and $F_1(H)$ the subset of all operators in $F(H)$ with rank less than or equal to 1. We call a subalgebra $\mathcal{A}$ of $L(H)$ standard provided $\mathcal{A}$ contains $F(H)$. A collection $\mathcal{L}$ of subspaces of $H$ is said to be a subspace lattice if it contains zero and $H$, and is complete in the sense that it is closed under the formation of arbitrary closed linear spans and intersections. A subspace lattice $\mathcal{N}$ is called a nest if it is a totally ordered subspace lattice. Given a nest $\mathcal{N}$, let alg $\mathcal{N} = \{T \in L(H) : TN \subseteq N, N \subseteq \mathcal{N}\}$. Alg $\mathcal{N}$ is said to be the nest algebra associated with $\mathcal{N}$. If $\mathcal{N}$ is a nest and $E \in \mathcal{N}$, then we define $E_- = \vee \{F \in \mathcal{N} : F \subseteq E\}$, and $E_+ = \wedge \{F \in \mathcal{N} : F \supseteq E\}$. If $e, f \in H$ we write $e^* \otimes f$ for the rank one operator $x \rightarrow (x,e)f$, whose norm is $\|e\|\|f\|$. If $\mathcal{N}$ is a nest, then by [8, Lemma 3.7], $e^* \otimes f \in \text{alg} \mathcal{N}$ if and only if there is an $E \in \mathcal{N}$...
such that $f \in E$ and $e \in (E_+)$. If $\mathcal{A}$ is a subalgebra of $L(H)$, then we say that $\mathcal{A}$ is a **triangular** operator algebra, if $\mathcal{A} \cap \mathcal{A}^*$ is a maximal abelian selfadjoint subalgebra of $L(H)$. If $\mathcal{J}$ is maximal triangular, and $\text{lat}\mathcal{A}$ is a maximal nest, then we say that $\mathcal{A}$ is **strongly reducible**. A derivation $\delta$ of an algebra $\mathcal{A}$ into an $\mathcal{A}$-bimodule $\mathcal{B}$ is a linear map satisfying $\delta(AB) = A\delta(B) + \delta(A)B$, for any $A, B \in \mathcal{A}$. A derivation $\delta$ is called $\mathcal{B}$-inner if there exists $T \in \mathcal{B}$, such that $\delta(A) = AT - TA$. When we say that a derivation $\delta : \mathcal{A} \rightarrow \mathcal{B}$ is inner, we mean $\mathcal{B}$-inner.

2. Derivations Let $\mathcal{N}$ be a nest. In the following, we consider the derivation from a subalgebra $\mathcal{A}$ of $L(H)$ containing all rank one operators of $\mathcal{N}$ into $L(H)$.

**Theorem 2.1.** If $\mathcal{N}$ is a nest such that $H_\perp \neq H$, $\mathcal{A}$ is a subalgebra of $L(H)$ containing $(\mathcal{N}) \cap F_1(H)$, and $\delta$ is a derivation from $\mathcal{A}$ into $L(H)$, then $\delta$ is inner.

**Proof.** Since $H_\perp \neq H$, for any $f^* \in (H_\perp)^\perp$, $f^* \neq 0$, we choose $\gamma$ in $H$ such that $f^*(\gamma) = 1$. For any $x$ in $H$, by [8, Lemma 3.7], it follows that $f^* \otimes x \in \mathcal{N}$. Now define

$$Tx = -\delta(f^* \otimes x)\gamma, \quad \text{for } x \in H.$$  

(2.1)

Now for $A$ in $\mathcal{A}$,

$$TAx = -\delta(f^* \otimes Ax)\gamma = -\delta(A)x - A\delta(f^* \otimes x)\gamma = -\delta(A)x + ATx.$$  

(2.2)

Hence for any $x \in H$, $-TAx + ATx = \delta(A)x$; thus

$$\delta(A) = AT - TA.$$  

(2.3)

It remains to show that $\delta$ is bounded.

Let $\lim_{n \to \infty} X_n = x$, and $\lim_{n \to \infty} TX_n = y$. Now for any rank one operator $A \in \mathcal{N}$, we have that $\delta(A)$ and $TA$ are bounded. It follows that $AT = \delta(A) + TA$ is bounded, and $\lim_{n \to \infty} ATX_n = ATx = Ay$. Since $\mathcal{A}$ contains all rank one operators of $\mathcal{N}$, and by [4, Proposition 3.8], every finite rank operator of $\mathcal{N}$ is a sum of some rank one operators of $\mathcal{A}$, we have, for any finite rank operator $B$ in $\mathcal{N}$, $BTx = By$. By [4, Theorem 3.11], choose a bounded net $\{B_\lambda\}$ of finite rank operators in $\mathcal{N}$ such that $\lim_\lambda B_\lambda = I$ in the strong operator topology. We have $TX = \gamma$. By the closed graph theorem, it follows that $T$ is bounded. \hfill \Box

**Corollary 2.2.** If $\mathcal{N}$ is a nest such that $0_\perp \neq 0$, and $\mathcal{A}$ is a subalgebra of $L(H)$ containing all rank one operators of $\mathcal{N}$, then every derivation $\delta$ from $\mathcal{A}$ into $L(H)$ is inner.

**Proof.** Let $\mathcal{N}^\perp = \{N^\perp : N \in \mathcal{N}\}$. Then $\mathcal{N}^\perp$ is a nest such that $H_\perp \neq H$. Since $\mathcal{A}^* = (\mathcal{N}^*)^*$, it follows that $\mathcal{A}^*$ contains all rank one operators of $\mathcal{A}^\perp$. Define $\delta^*(A = (\delta(A^*))^*)$ for any $A$ in $\mathcal{A}^*$.

It is easy to prove that $\delta^*$ is a derivation from $\mathcal{A}^*$ into $L(H)$. By Theorem 2.1, we have that $\delta^*$ is inner. It follows that $\delta$ is inner. \hfill \Box

We now consider a nest $\mathcal{N}$ such that $H_\perp = H$. 


\textbf{Lemma 2.3.} Let $\mathcal{N}$ be a nest, $E_1, E_2 \in \mathcal{N}$ and $E_1 \subseteq E_2$. If $T$ is a linear map from $E_2$ into $H$ such that $ST = TS$ on $E_2$ for any rank one operator $S$ of alg $\mathcal{N}$, then there exists a $\lambda$ such that $Tx = \lambda x$, for any $x \in E_1$.

\textbf{Proof.} For $x \in E_1$, choose $y \in E_2 - E_1$ such that $\|y\| = 1$. Since $y^* \otimes x \in \text{alg } \mathcal{N}$, by hypothesis

$$Ty^* \otimes x(y) = y^* \otimes xTy = (Ty, y)x. \quad (2.4)$$

Since every one-dimensional subspace of $L(E_2, H)$ is reflexive, it follows that there exists $\lambda$ such that $T = \lambda I$. \hfill $\square$

\textbf{Lemma 2.4.} Let $\mathcal{N}$ be a nest such that $H_- = H$, and let $M = \cup \{ N : N \not\subseteq H, \ N \in \mathcal{N} \}$. Then there exists a linear map $T$ from $M$ into $H$ such that $\delta(A)x = (AT - TA)x$, for any $x \in M$.

\textbf{Proof.} Since $H_- = H$, we may choose an increasing sequence $\{ P_i \} \subseteq \mathcal{N}$ such that $P_i \to I$ in the strong operator topology. Also choose $f^* \in P_i^\ast$, and $y \in H$, such that $\|f^*\| = 1$, $f^*(y) = 1$, and $\|y\| \leq 2$. Define,

$$T_ix = -\delta(f^* \otimes x)y \quad \text{for } x \in P_i. \quad (2.5)$$

Using an argument similar to the proof of Theorem 2.1, we may prove that for $A$ in $\mathcal{A}$, $\delta(A)x = (AT_i - T_iA)x$ for $x \in P_i$. If $j \geq i$, then for $x \in P_i$, $(AT_i - T_iA)x = (AT_j - T_jA)x$. Hence

$$A(T_i - T_j)x = (T_i - T_j)Ax, \quad \text{for } x \in P_i. \quad (2.6)$$

By Lemma 2.3, we have $T_j - T_i = \lambda_{ij}$ on $P_{i-1}$. Now for $j > i$, let $\tilde{T}_j = T_1 + \lambda_{i,j}$. We have, for $k > j > 2$, $\tilde{T}_jx = \tilde{T}_kx$ for all $x \in P_{j-1}$. Now for any $x \in \cup \{ P_i \} = \cup \{ N : N \not\subseteq H, \ N \in \mathcal{N} \}$, choose a $j > 2$ such that $x \in P_j$ and let $Tx = \tilde{T}_jx$. Then, $T$ is well defined and for $x$ in $M$, $\delta(A)x = (AT - TA)x$. \hfill $\square$

\textbf{Remark 2.5.} Using the idea in the proof of Theorem 2.1, we can prove that in Lemma 2.3, $T_i$ is a bounded operator from $P_i$ into $H$.

\textbf{Theorem 2.6.} If $\mathcal{N}$ is a nest, $\mathcal{A}$ is a subalgebra of $L(H)$ containing all rank one operators of alg $\mathcal{N}$, and $\delta$ is a norm continuous derivation from $\mathcal{A}$ into $L(H)$, then $\delta$ is inner.

\textbf{Proof.} If $\mathcal{N}$ satisfies $H_- \neq H$, then by Theorem 2.1, we get that $\delta$ is inner. If $\mathcal{N}$ satisfies $H_- = H$, then by Lemma 2.4, there exists a linear map $T$ such that

$$\delta(A)x = (AT - TA)x \quad \text{for any } x \in M = \cup \{ N : N \not\subseteq H, \ N \in \mathcal{N} \}. \quad (2.7)$$

By (2.5) and the boundedness of $\delta$, it follows that $\|T_ix\| \leq 2\|\delta\|\|x\|$. Since $|\lambda_{ij}| \leq \|T_i\| + \|T_j\| \leq 4\|\delta\|$, it follows that $\|T\| \leq 6\|\delta\|$. Thus $T$ is bounded on $M$. Let $\tilde{T}$ be the unique bounded extension of $T$ to $H$. Then $\tilde{T}$ is bounded and for $A$ in $\mathcal{A}$, $\delta(A) = A\tilde{T} - \tilde{T}A$. \hfill $\square$
Theorem 2.7. Let $\mathcal{N}$ be a nest satisfying $H_- = H$. If there exists a nontrivial projection $P \in \mathcal{N}$, such that $P \in \mathcal{A}$, and $\delta$ is a derivation from $\mathcal{A}$ into $L(H)$, then $\delta$ is inner.

Proof. As in the proof of Lemma 2.4, we choose $P_1 = P$. Let $H = P \oplus P^\perp$. Then $T$ can be decomposed as

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}.$$  \hfill (2.8)

Let $Q = \cup \{N - P : P \subseteq N \in \mathcal{N}, N \neq H\}$, $T_{12} : Q \rightarrow P$, $T_{22} : Q \rightarrow Q$.

By the definition of $T$, $T_{11}$ and $T_{21}$ are bounded. We now prove that $T_{12}$ and $T_{22}$ are bounded. Since $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ in $\mathcal{A}$, we have that $\delta(A) = \begin{pmatrix} 0 & T_{12} \\ -T_{21} & 0 \end{pmatrix}$ holds on $M$. Since $\delta(A)$ is bounded, it follows that $T_{12}$ is bounded. Now, for any rank one operator $A \in L(H)$, we have $PA(1 - P) \in \mathcal{A}$. Hence,

$$\delta(PA(1 - P)) = \begin{pmatrix} PA(1 - P) & PA(1 - P)T_{22} - T_{11} \\ 0 & -T_{21}PA(1 - P) \end{pmatrix}.$$  \hfill (2.9)

holds on $M$. Since $\delta(PA(1 - P))$ is bounded, it follows that $PA(1 - P)T_{22}$ is bounded. Hence for any $f^* \in P^\perp$ and $e \in P$, $e \neq 0$, $f^* \otimes eT_{22}$ is bounded on $Q$. Thus there exists $c$ such that $|f^*(T_{22}x)| \leq c$, for any $x \in Q$ and $\|x\| \leq 1$. By the uniform boundedness theorem, we have that $\{\|T_{22}x\| : \|x\| \leq 1\}$ is bounded. Hence $T_{22}$ is bounded. As in Theorem 2.6, there exists a bounded extension $\tilde{T}$ of $T$ to $H$ such that for $A$ in $\mathcal{A}$, $\delta(A) = A\tilde{T} - \tilde{T}A$.  

3. Applications. In this section, we apply the results above to some special subalgebras of $L(H)$. If $A \supseteq F(H)$, then by Theorem 2.1, we have the following corollaries.

Corollary 3.1 [1]. Every derivation from a standard operator algebra into $L(H)$ is inner.

Corollary 3.2 [2]. If $\delta$ is a derivation from alg $\mathcal{N}$ into itself, then $\delta$ is inner.

Proof. By Theorems 2.1 and 2.7, we have that there is $T$ in $L(H)$ such that for any $A$ in $\mathcal{A}$, $\delta(A) = AT - TA$. Now we prove that $T$ is in alg $\mathcal{N}$. Now for any $P$ in $\mathcal{N}$, since $\delta(P) = PT - TP$ in alg $\mathcal{N}$, we have that $(I - P)\delta(P)P = 0 = -(I - P)TP$. This completes the proof.  

Let $\mathcal{B}$ be a subalgebra of $L(H)$, and let $\mathcal{F}$ be any subset of $L(H)$. We denote by $C(\mathcal{B}, \mathcal{F})$ the collection, $\{T \in L(H) : AT - TA \in \mathcal{F}, \forall A \in \mathcal{B}\}$.

Lemma 3.3 [6]. Let $\mathcal{B}$ be a nest algebra and $\mathcal{F}$ be an ideal in $L(H)$. Then $C(\mathcal{B}, \mathcal{F}) = CI + \mathcal{F}$.

Using this lemma and Theorem 2.7, we easily prove the following result.

Corollary 3.4. If $\mathcal{B}$ is an algebra containing alg $\mathcal{N}$, then any derivation $\delta : \mathcal{B} \rightarrow C_p$ is inner for $1 \leq p \leq \infty$.

Corollary 3.5. If $\mathcal{B}$ is a triangular operator algebra containing every rank one operator in alg $\mathcal{N}$, then every derivation $\delta$ from $\mathcal{B}$ into $L(H)$ is inner.
PROOF. Suppose \( \tilde{N} \) is a maximal nest containing \( \mathcal{N} \). By hypothesis we have that \( B \supseteq (\text{alg} \mathcal{N}) \cap F_1(\mathcal{H}) \). Since \( \mathcal{B} \) contains all rank one operators of \( \text{alg} \mathcal{N} \), we have that \( \text{lat} \mathcal{B} \subseteq \mathcal{N} \). By [5, Theorem 4], it follows that \( \text{lat} \mathcal{B} = \tilde{N} = \mathcal{N} \). Since \( \mathcal{B} \) is a triangular operator algebra, it follows \( \tilde{N} \subseteq \mathcal{B} \).

If \( H_+ \neq H \), then by Theorem 2.1, we have that \( \delta \) is inner.

If \( H_+ = H \), \( \mathcal{N} \subseteq \mathcal{B} \), and \( \mathcal{N} \) is a maximal nest, by Theorem 2.7, it follows that \( \delta \) is inner.

\[ \square \]

REMARK 3.6. By Corollary 3.1, it follows that every derivation \( \delta : F(H) \to L(H) \) is inner. However if \( \mathcal{B} \) is a unital algebra containing \( F(H) \) and \( \mathcal{B} \neq L(H) \), then there is a derivation from \( F(H) \) into \( \mathcal{B} \) that is not inner, e.g., \( \delta = \delta_T \) with \( T \notin \mathcal{B} \). Also if \( \mathcal{A} = K(H) + CI \), and \( T \notin \mathcal{A} \), then \( \delta_T : \mathcal{A} \to \mathcal{A} \) is a derivation that is not inner, but \( \mathcal{A} \) contains all rank one operators of \( L(H) \).

By [8, Lemma 5.2], we know that if \( \mathcal{B} \) is a strongly reducible maximal triangular algebra, then \( \text{lat} \mathcal{B} \) is a nest and \( \mathcal{B} \) contains all rank one operators of \( \text{alg} \text{lat}(\mathcal{B}) \). Hence by Corollary 3.5 and Theorem 2.7, we have the following result.

COROLLARY 3.7. Every derivation from a strongly reducible maximal triangular algebra into \( L(H) \) is inner.

ACKNOWLEDGEMENTS. The authors thank Professors Don Hadwin and Rita Hibschweiler for their help with the paper. The authors would also like to thank the referee for useful suggestions.

REFERENCES


Jiankui Li: Department of Mathematics, Hunan Normal University, Hunan 410081, China
Current address: Department of Mathematics, University of New Hampshire, Durham, NH 03824, USA
E-mail address: jkli@spicerack.sr.unh.edu

Hemant Pendharkar: Department of Mathematics and Computer Science, Elizabeth City State University, Elizabeth City, NC 27909, USA
E-mail address: hemant@umfort.cs.ecsu.edu