MULTIPLIERS ON SOME WEIGHTED L^p -SPACES

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ABSTRACT. Let G be a locally compact abelian group with Haar measure dx , and let ω be a symmetric Beurling weight function on G (Reiter, 1968). In this paper, using the relations between p_i and q_i , where $1 < p_i, q_i < \infty$, $p_i \neq q_i$ ($i = 1, 2$), we show that the space of multipliers from $L_\omega^{p_1}(G)$ to the space $S(q'_1, q'_2, \omega^{-1})$, the space of multipliers from $L_\omega^{p_1}(G) \cap L_\omega^{p_2}(G)$ to $L_\omega^q(G)$ and the space of multipliers $L_\omega^{p_1}(G) \cap L_\omega^{p_2}(G)$ to $S(q'_1, q'_2, \omega^{-1})$.

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1. Introduction. Let G be a locally compact abelian group with Haar measure dx . In this paper, $C_0(G)$ denotes the space of all continuous complex-valued functions on G with compact support. Let $y \in G$. Then the translation operator τ_y is defined by

$$\tau_y f = f(y). \quad (1.1)$$

For a Beurling weight function on G (see Reiter [6]), i.e., a continuous function ω satisfying $\omega(x) \geq 1$ and $\omega(x+y) \leq \omega(x)\omega(y)$ for all $x, y \in G$.

We set, for $1 \leq p \leq \infty$, $L_\omega^p(G) = \{f \mid f\omega \in L^p(G)\}$. This is a Banach space under the norm

$$\|f\|_{p,\omega} = \left(\int_G |f(x)\omega(x)|^p dx \right)^{1/p}. \quad (1.2)$$

$L_\omega^1(G)$ is a Banach algebra with respect to convolution under the norm $\|\cdot\|_{1,\omega}$. It is called Beurling algebra. If $(1/p) + (1/p') = 1$, then the conjugate space of $L_\omega^p(G)$ is $L_{\omega^{-1}}^{p'}(G)$.

Let $1 < p_i < \infty$ ($i = 1, 2$) and let $S(p_1, p_2, \omega)$ be the set of all complex-valued functions g which can be written as

$$g = g_1 + g_2 \quad \text{with } (g_1, g_2) \in L_\omega^{p_1}(G) \times L_\omega^{p_2}(G). \quad (1.3)$$

We define a norm on $S(p_1, p_2, \omega)$ by

$$\|g\|_s = \inf (\|g_1\|_{p_1,\omega} + \|g_2\|_{p_2,\omega}), \quad (1.4)$$

where the infimum is taken over all such decompositions of g . $S(p_1, p_2, \omega)$ is a Banach space under this norm (see Liu and Wang [4]).

Similarly, if $D(p_1, p_2, \omega)$ denotes the set of all complex-valued functions defined on G which are in $L_\omega^{p_1}(G) \cap L_\omega^{p_2}(G)$, we introduce a norm by

$$\|f\|_{D_p} = \max (\|f\|_{p_1,\omega}, \|f\|_{p_2,\omega}). \quad (1.5)$$

Then, $D(p_1, p_2, \omega)$ is also a Banach space with the norm $\|\cdot\|_{D_p}$.

It is not hard to see that $D(p_1, p_2, \omega)$ is a Banach $L_\omega^1(G)$ module. It is known that $D(p_1, p_2, \omega)$ and $S(p_1, p_2, \omega)$ are reflexive Banach spaces and the following duality relations hold:

$$\begin{aligned} D(p_1, p_2, \omega)^* &\cong S(p'_1, p'_2, \omega^{-1}), \\ D(p'_1, p'_2, \omega^{-1})^* &\cong S(p_1, p_2, \omega), \end{aligned} \quad (1.6)$$

where $(1/p_i) + (1/p'_i) = 1$, $i = 1, 2$ (see Murthy-Unni [5] and Liu and Wang [4]).

Let $f \in L_\omega^{q_1}(G) \cap L_\omega^{q_2}(G)$ with $1 \leq q_1 \leq q_2 \leq \infty$. Then, $f \in L_\omega^q(G)$ for all $q_1 \leq q \leq q_2$ and

$$\|f\|_{q, \omega} \leq \|f\|_{q_1, \omega}^\alpha \|f\|_{q_2, \omega}^{1-\alpha}, \quad (1.7)$$

where $1/q = (\alpha/q_1) + (1-\alpha)/q_2$, $0 \leq \alpha \leq 1$ (see Brezis [1]).

Now, we define the space $K(p, q, q_1, q_2, \omega)$ to be set of all functions h which can be written in the form

$$h = \sum_{i=1}^{\infty} f_i^* g_i, \quad (1.8)$$

where $f_i \in C_c(G) \subset L_\omega^p(G)$ and $g \in D(q_1, q_2, \omega)$ with $\sum_{i=1}^{\infty} \|f_i\|_{p, \omega} \|g_i\|_D < \infty$. We define a norm on $K(p, q, q_1, q_2, \omega)$ by

$$\|h\| = \inf \left(\sum_{i=1}^{\infty} \|f_i\|_{p, \omega} \|g_i\|_{D_q} \right), \quad (1.9)$$

where the infimum is taken over all such representations of h . Then $K(p, q, q_1, q_2, \omega)$ is a Banach space in this norm. Since

$$\|f^* g\|_{r, \omega} \leq \|f\|_{p, \omega} \|g\|_{q, \omega} \leq \|f\|_{p, \omega} \|g\|_{D_q} \quad (1.10)$$

for $f \in C_c(G) \subset L_\omega^p(G)$, $g \in L_\omega^{q_1}(G) \cap L_\omega^{q_2}(G)$, where $(1/p) + (1/q) \geq 1$, $(1/r) = (1/p) + (1/q) - 1$, and condition (1.7) holds, it follows that $K(p, q, q_1, q_2, \omega) \subset L_\omega^r(G)$ and that the topology on $K(p, q, q_1, q_2, \omega)$ is not weaker than the topology induced from $L_\omega^r(G)$.

We say that T is a multiplier from $L_\omega^p(G)$ to $S(q'_1, q'_2, \omega^{-1})$ if T is a bounded linear operator on $L_\omega^p(G)$ which commutes with translation. The space of all multipliers from $L_\omega^p(G)$ to $S(q'_1, q'_2, \omega^{-1})$ is denoted by $M[L_\omega^p(G), S(q'_1, q'_2, \omega^{-1})]$.

2. Multipliers from $L_\omega^p(G)$ to $S(q'_1, q'_2, \omega^{-1})$. We have the following.

THEOREM 2.1. *Let G be a locally compact abelian group and let ω be a symmetric Beurling weight function. If condition (1.7) is satisfied and $(1/p) + (1/q) \geq 1$, $(1/p) + (1/q) - 1 = 1/r$, then the space of multipliers $M[L_\omega^p(G), S(q'_1, q'_2, \omega^{-1})]$ is isometrically isomorphic to the dual $K(p, q, q_1, q_2, \omega)^*$ of $K(p, q, q_1, q_2, \omega)$.*

PROOF. For any $T \in M[L_\omega^p(G), S(q'_1, q'_2, \omega^{-1})]$, define

$$t(h) = \sum_{i=1}^{\infty} T f_i^* g_i(0) \quad (2.1)$$

for $h = \sum_{i=1}^{\infty} f_i^* g_i$ in $K(p, q, q_1, q_2, \omega)$. First, we show that t is well defined. To this end, it is sufficient to show that if $h = \sum_{i=1}^{\infty} f_i^* g_i = 0$ in $K(p, q, q_1, q_2, \omega)$ and $\sum_{i=1}^{\infty} \|f_i\|_{p, \omega} \|g_i\|_{D_q} < \infty$, then $\sum_{i=1}^{\infty} T f_i^* g_i(0) = 0$.

It is known that $L^p_{\omega}(G)$ has approximate identities bounded in $L^1_{\omega}(G)$ with compactly supported (see Murthy-Unni [5]). Let $(\phi_{\alpha})_{\alpha \in I}$ be an approximate identity for $L^p_{\omega}(G)$ with $\|\phi_{\alpha}\|_1 = 1$ and $\|\phi_{\alpha}\|_{1, \omega} \leq K$ ($K > 0$). Then, for each $f \in L^p_{\omega}(G)$, we have

$$\lim_{\alpha} \|\phi_{\alpha}^* f - f\|_{p, \omega} = 0. \quad (2.2)$$

Therefore, using (2.2) and the fact that T is a multiplier for all $g \in L^{q_1}_{\omega}(G) \cap L^{q_2}_{\omega}(G)$, we obtain

$$|T(\phi_{\alpha}^* f_i)^* g_i(0) - T f_i^* g_i(0)| \leq \|T\| \|\phi_{\alpha}^* f_i - f_i\|_{p, \omega} \|g_i\|_{D_q} \rightarrow 0, \quad (2.3)$$

so that

$$\lim_{\alpha} T(\phi_{\alpha}^* f_i)^* g_i(0) = T f_i^* g_i(0). \quad (2.4)$$

Also, for each $\phi_{\alpha} \in C_c(G)$ and $f_i \in C_c(G)$, we have

$$T(\phi_{\alpha}^* f_i) = T \phi_{\alpha}^* f_i. \quad (2.5)$$

(see Larsen [2]).

Since $u = \sum_{i=1}^{\infty} f_i^* g_i = 0$ and the series $\sum_{i=1}^{\infty} f_i^* g_i$ converges uniformly and using equality (2.5), we get

$$\begin{aligned} \sum_{i=1}^{\infty} T(\phi_{\alpha}^* f_i)^* g_i(0) &= \sum_{i=1}^{\infty} \int_G (T \phi_{\alpha})(-y) (f_i^* g_i)(y) dy \\ &= \int_G (T \phi_{\alpha})(-y) \sum_{i=1}^{\infty} (f_i^* g_i)(y) dy = 0. \end{aligned} \quad (2.6)$$

We show that $\sum_{i=1}^{\infty} T(\phi_{\alpha}^* f_i)^* g_i(0)$ converges uniformly with respect to α .

$$\begin{aligned} \left| \sum_{i=1}^{\infty} T(\phi_{\alpha}^* f_i)^* g_i(0) \right| &\leq \sum_{i=1}^{\infty} \|T(\phi_{\alpha}^* f_i)\|_s \|g_i\|_{D_q} \\ &\leq \|T\| \sum_{i=1}^{\infty} \|\phi_{\alpha}^* f_i\|_{p, \omega} \|g_i\|_{D_q} \\ &= \|T\| \sum_{i=1}^{\infty} \|f_i\|_{p, \omega} \|g_i\|_{D_q} < \infty. \end{aligned} \quad (2.7)$$

The convergence of $\sum_{i=1}^{\infty} T(\phi_{\alpha}^* f_i)^* g_i(0)$ is uniform with respect to α . Hence,

$$\lim_{\alpha} \sum_{i=1}^{\infty} T(\phi_{\alpha}^* f_i)^* g_i(0) = \sum_{i=1}^{\infty} T f_i^* g_i(0) = 0 \quad (2.8)$$

using (2.4) and (2.6). Thus, t is well defined.

It is obvious that the mapping $T \rightarrow t$ is linear. Now we show that it is an isometry. In fact,

$$|t(u)| = \left| \sum_{i=1}^{\infty} T f_i^* g_i(0) \right| \leq \sum_{i=1}^{\infty} \|T f_i\|_s \|g_i\|_{D_q} \leq \|T\| \sum_{i=1}^{\infty} \|f_i\|_{p,\omega} \|g_i\|_{D_q} \quad (2.9)$$

implies that

$$|t(u)| \leq \|T\| \|u\|. \quad (2.10)$$

Hence, $\|t\| \leq \|T\|$. On the other hand,

$$\begin{aligned} \|T\| &= \sup \left\{ |T f^* g(0)| : \|f\|_{p,\omega} \leq 1, \|g\|_{D_q} \leq 1 \right\} \\ &= \sup \left\{ |t(f^* g)| : \|f\|_{p,\omega} \leq 1, \|g\|_{D_q} \leq 1 \right\} \leq \|t\|. \end{aligned} \quad (2.11)$$

Therefore, $\|t\| = \|T\|$.

Finally, we show that the mapping $T \rightarrow t$ is onto. Suppose that $t \in K(p, q, q_1, q_2, \omega)^*$ and that $f \in C_c(G) \subset L_\omega^p(G)$. Define, for all $g \in L_\omega^{q_1}(G) \cap L_\omega^{q_2}(G)$,

$$g \mapsto t(f^* g). \quad (2.12)$$

Then,

$$|t(f^* g)| \leq \|t\| \|f^* g\| \leq \|t\| \|f\|_{p,\omega} \|g\|_{D_q} \quad (2.13)$$

implies that this mapping gives a bounded linear functional on $L_\omega^{q_1}(G) \cap L_\omega^{q_2}(G)$. Hence, there exists a unique element, denoted by Tf , in $S(q'_1, q'_2, \omega^{-1})$ such that

$$T f^* g(0) = t(f^* g) \quad (2.14)$$

for all $g \in L_\omega^{q_1}(G) \cap L_\omega^{q_2}(G)$, and

$$|T f^* g(0)| \leq \|t\| \|f\|_{p,\omega} \|g\|_{D_q}. \quad (2.15)$$

Therefore,

$$\|T f\|_s \leq \|t\| \|f\|_{p,\omega} < \infty. \quad (2.16)$$

Hence, T is a bounded operator from $C_c(G)$ into $S(q'_1, q'_2, \omega^{-1})$. Clearly, T is linear. Since $C_c(G)$ is dense $L_\omega^p(G)$. It can be extended uniquely as a bounded linear operator on $L_\omega^p(G)$. We have to prove that this extended T is a multiplier. Let $y \in G$ and $f \in C_c(G) \subset L_\omega^p(G)$. If $g \in L_\omega^{q_1}(G) \cap L_\omega^{q_2}(G)$, then

$$\tau_y T f^* g(0) = T f^* \tau_y g(0) = t(f^* \tau_y g) = t(\tau_y f^* g) = T \tau_y f^* g(0) \quad (2.17)$$

holds for all functions g in $L_\omega^{q_1}(G) \cap L_\omega^{q_2}(G)$. Hence,

$$\tau_y T f = T \tau_y f. \quad (2.18)$$

Thus, T belongs to $M[L_\omega^p(G), S(q'_1, q'_2, \omega^{-1})]$ and our assertion is proved. $\square$

3. Multipliers from $L_\omega^{p_1} \cap L_\omega^{p_2}(G)$ to $L_\omega^q(G)$. Let $f \in L_\omega^{p_1}(G) \cap L_\omega^{p_2}(G)$, with $1 < p_1 \leq p_2 < \infty$, then $f \in L_\omega^p(G)$ for all $p_1 \leq p \leq p_2$ and

$$\|f\|_{p,\omega} \leq \|f\|_{p_1,\omega}^\alpha \|f\|_{p_2,\omega}^{1-\alpha}, \quad (3.1)$$

where $1/p = (\alpha/p_1) + (1-\alpha)/p_2$, $0 \leq \alpha \leq 1$.

Now, we define the space $K(p_1, p_2, p, q, \omega)$ to be set of all functions h which can be written in the form

$$h = \sum_{i=1}^{\infty} f_i^* g_i, \quad (3.2)$$

where $f_i \in C_c(G) \subset L_\omega^{p_1} \cap L_\omega^{p_2}(G)$ and $g_i \in L_{\omega^{-1}}^{q'}(G)$ with $\sum_{i=1}^{\infty} \|f_i\|_{D_p} \|g_i\|_{q',\omega^{-1}} < \infty$. We define a norm on $K(p_1, p_2, p, q, \omega)$ by

$$\|h\| = \inf \left(\sum_{i=1}^{\infty} \|f_i\|_{D_p} \|g_i\|_{q',\omega^{-1}} \right), \quad (3.3)$$

where the infimum is taken over all such representations of h . Then $K(p_1, p_2, p, q, \omega)$ is a Banach space in this norm. Since

$$\|f^* g\|_{r,\omega^{-1}} \leq \|f\|_{p,\omega} \|g\|_{q',\omega^{-1}} \leq \|f\|_{D_p} \|g\|_{q',\omega^{-1}} < \infty \quad (3.4)$$

for $f \in C_c(G) \subset L_\omega^{p_1}(G) \cap L_\omega^{p_2}(G)$ and $g \in L_{\omega^{-1}}^{q'}(G)$, where condition (3.1) holds and $1 < p \leq q < \infty$, $1/r = (1/p) - (1/q)$, it follows that $K(p_1, p_2, p, q, \omega) \subset L_{\omega^{-1}}^{r'}(G)$.

THEOREM 3.1. *Let G be a locally compact abelian group and let ω be a symmetric Beurling weight function under condition (3.1) and $1 < p \leq q < \infty$, $1/r = (1/p) - (1/q)$, then the space of multipliers $M[L_\omega^{q_1}(G) \cap L_\omega^{q_2}(G), L_\omega^q(G)]$ is isometrically isomorphic to the dual $K(p_1, p_2, p, q, \omega)^*$ of $K(p_1, p_2, p, q, \omega)$.*

PROOF. Using the same method in the proof of Theorem 2.1, we can show our assertion. $\square$

4. Multipliers from $L_\omega^{p_1}(G) \cap L_\omega^{p_2}(G)$ to the space $S(q'_1, q'_2, \omega^{-1})$. Suppose that $(1/p_i) + (1/q_i) \geq 1$, $1 < p_i$, $q_i < \infty$ and $(1/p_i) + (1/q_i) - 1 = 1/r_i$ ($i = 1, 2$). Let $L_\omega^{r_1, r_2}(G)$ denote the set of all complex-valued functions defined on G which are in $L_\omega^{r_1}(G) \cap L_\omega^{r_2}(G)$. We introduce a norm by

$$\|f\|_\omega^{r_1, r_2} = \max(\|f\|_{r_1, \omega}, \|f\|_{r_2, \omega}). \quad (4.1)$$

Then $L_\omega^{r_1, r_2}(G)$ is also a Banach space with norm $\|\cdot\|_\omega^{r_1, r_2}$ (see Liu and Wang [3]).

To obtain the space of multipliers from $L_\omega^{p_1}(G) \cap L_\omega^{p_2}(G)$ to $S(q'_1, q'_2, \omega^{-1})$ as a dual space, we define the space $K(p_1, p_2, q_1, q_2, \omega)$ to be the set of all functions h which can be written in the form

$$h = \sum_{i=1}^{\infty} f_i^* g_i, \quad (4.2)$$

where $f_i \in C_c(G) \subset L_\omega^{p_1}(G) \cap L_\omega^{p_2}(G)$ and $g_i \in L_\omega^{q_1}(G) \cap L_\omega^{q_2}(G)$ with $\sum_{i=1}^{\infty} \|f_i\|_{D_p} \|g_i\|_{D_q} < \infty$

∞ . It is not hard to see that $C_c(G)$ is dense in $L_\omega^{p_1}(G) \cap L_\omega^{p_2}(G)$. Define a norm $h \rightarrow \|h\|$ by

$$\|h\| = \inf \left(\sum_{i=1}^{\infty} \|f_i\|_{D_p} \|g_i\|_{D_q} \right), \quad (4.3)$$

where the infimum is taken over all such representations of h . It is easy to verify that $\|\cdot\|$ defines a norm on $K(p_1, p_2, q_1, q_2, \omega)$ and that the latter is a Banach space.

Now, let $f \in C_c(G) \subset L_\omega^{p_1}(G) \cap L_\omega^{p_2}(G)$ and $g \in L_\omega^{q_1}(G) \cap L_\omega^{q_2}(G)$. It follows that $f^*g \in L_\omega^{r_1}(G)$,

$$\|f^*g\|_{r_1, \omega} \leq \|f\|_{p_1, \omega} \|g\|_{q_1, \omega} \leq \|f\|_{D_p} \|g\|_{D_q} \quad (4.4)$$

and $f^*g \in L_\omega^{r_2}(G)$,

$$\|f^*g\|_{r_2, \omega} \leq \|f\|_{p_2, \omega} \|g\|_{q_2, \omega} \leq \|f\|_{D_p} \|g\|_{D_q} \quad (4.5)$$

so that

$$\|f^*g\|_\omega^{r_1, r_2} \leq \|f\|_{D_p} \|g\|_{D_q}. \quad (4.6)$$

From this, it is clear that $K(p_1, p_2, q_1, q_2, \omega) \subset L_\omega^{r_1, r_2}(G)$ and that the topology on $K(p_1, p_2, q_1, q_2, \omega)$ is not weaker than the topology induced by $L_\omega^{r_1, r_2}(G)$.

THEOREM 4.1. *Let G be a locally compact abelian group and let ω be a symmetric Beurling weight function. If $(1/p_i) + (1/q_i) \geq 1$, $(1/p_i) + (1/q_i) - 1 = (1/r_i)$, $i = 1, 2$, then the space of multipliers $M[L_\omega^{p_1}(G) \cap L_\omega^{p_2}(G), S(q'_1, q'_2, \omega)]$ is isometrically isomorphic to $K(p_1, p_2, q_1, q_2, \omega)^*$, the dual space of $K(p_1, p_2, q_1, q_2, \omega)$.*

PROOF. Use the same method employed in the proof of Theorem 2.1. □

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