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A UNIQUENESS THEOREM FOR BOEHMIANS OF ANALYTIC TYPE

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ABSTRACT. The following uniqueness theorem is proven for a class of generalized functions. If F is a Boehmian of analytic type and $F = 0$ on some open arc Ω, then $F \equiv 0$.

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1. Introduction. It is well known that if $f(z)$ is a bounded analytic function in the unit disk D, then it has a radial limit $F(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$ almost everywhere on the unit circle T. Moreover, if $F(e^{i\theta}) = 0$ on a set $E \subseteq T$ of positive measure, then $f(z)$ is identically zero. This property is called the uniqueness property, since bounded analytic functions in D are uniquely determined by their boundary values $F(e^{i\theta})$ on E.

Similar results can be obtained for a larger class of analytic functions in D, where each function $f(z)$ in this class corresponds to a Beurling distribution F on T. Moreover, if $F = 0$ (in the sense of distributions) on an open arc, then $f(z)$ is identically zero [6].

Thus, let A be a space of generalized functions on T. Two questions arise:

(1) Does the space A have a uniqueness property?

(2) What type of function in the unit disk has an element of A as a boundary value?

In this paper we consider a space of generalized functions on T called Boehmians. The space of Boehmians is a generalization of Schwartz distributions as well as Beurling distributions. In [3, 4] the author shows, among other things, that there are Boehmians which are not hyperfunctions and hyperfunctions which are not Boehmians. For other results, which also includes a more general construction of spaces of Boehmians, see Mikusiński [1].

In this paper we show that a uniqueness property holds for a subspace of Boehmians known as Boehmians of analytic type.

2. Preliminaries. Let $C(T)$ denote the collection of continuous complex-valued functions on the unit circle T. We make no distinction between a function on T and a 2π-periodic function on the real line $\mathbb{R}$.

The convolution of two functions $f, g \in C(T)$, denoted by $f * g$, is given by

$$(f * g)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)g(t)\,dt. \tag{2.1}$$

A sequence of continuous nonnegative functions $\{\delta_n\}$ will be called a *delta sequence* if

(i) $1/2\pi \int_{-\pi}^{\pi} \delta_n(x)dx = 1$ for all $n \in \mathbb{N}$,
(ii) $\operatorname{supp} \delta_n \subseteq (-\varepsilon_n, \varepsilon_n)$, where $\varepsilon_n \to 0$ as $n \to \infty$.

The collection of delta sequences will be denoted by Δ.

Let $C^N(T)$ denote the collection of sequences of elements from $C(T)$ and $\mathscr{A} \subseteq C^N(T) \times \Delta$ be defined as follows:

$$\mathscr{A} = \{(\{f_n\}, \{\delta_n\}) : f_k * \delta_n = f_n * \delta_k \text{ for all } n, k \in \mathbb{N}\}. \tag{2.2}$$

Two elements $(\{f_n\}, \{\delta_n\})$ and $(\{g_n\}, \{\sigma_n\})$ of $\mathscr{A}$ are said to be equivalent if $f_k * \sigma_n = g_n * \delta_k$ for all $n, k \in \mathbb{N}$. A straightforward calculation shows that this is an equivalence relation on $\mathscr{A}$. The equivalence classes are called periodic Boehmians.

Define

$$\beta = \left\{ \left[\frac{\{f_n\}}{\{\delta_n\}} \right] : (\{f_n\}, \{\delta_n\}) \in \mathscr{A} \right\}. \tag{2.3}$$

For convenience a typical element of β will be written as $F = f_n/\delta_n$.

By defining a natural addition, multiplication, and scalar multiplication on β, i.e.,

$$\frac{f_n}{\delta_n} + \frac{g_n}{\sigma_n} = \frac{f_n * \sigma_n + g_n * \delta_n}{\delta_n * \sigma_n}, \qquad \frac{f_n}{\delta_n} * \frac{g_n}{\sigma_n} = \frac{f_n * g_n}{\delta_n * \sigma_n}, \qquad \alpha \frac{f_n}{\delta_n} = \frac{\alpha f_n}{\delta_n}, \tag{2.4}$$

where α is a complex number, β becomes a commutative algebra with identity $\delta = \delta_n/\delta_n$.

3. Boehmians of analytic type. The nth Fourier coefficient for a function $f \in C(T)$ is defined in the usual way,

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx, \quad n \in \mathbb{Z}. \tag{3.1}$$

Definition 3.1. Let $F = f_n/\delta_n \in \beta$. The nth Fourier coefficient of F, denoted by $\hat{F}(n)$, is defined by $\hat{F}(n) = \lim_{k\to\infty} \hat{f}_k(n)$.

The limit in the above definition always exists and is independent of the representative used for F (see [2]).

The sequence of Fourier coefficients for a Boehmian can behave quite differently than the sequence of Fourier coefficients of a distribution or hyperfunction. For example, the sequence of Fourier coefficients for a Boehmian cannot grow unrestricted, however a subsequence may grow without any restrictions (see [4]).

A Boehmian F is said to be of *analytic type* if $\hat{F}(n) = 0$, for $n = -1, -2, \ldots$.

Definition 3.2. A Boehmian F is said to be zero on an open set Ω, denoted by $F = 0$ on Ω, if there exists a delta sequence $\{\delta_n\}$ such that $F * \delta_n \in C(T)$ for all $n \in \mathbb{N}$ and $F * \delta_n \to 0$ uniformly on compact subsets of Ω as $n \to \infty$.

As an example, consider the Boehmian $\delta = \delta_n/\delta_n$. Then, $\delta = 0$ on $\Omega = \{x : 0 < |x| < 2\pi\}$.

It appears that the above definition may depend on the delta sequence $\{\delta_n\}$. However it is not difficult to show that if $\{\sigma_n\}$ is a delta sequence such that $F * \sigma_n \in C(T)$

for all $n \in \mathbb{N}$, then the sequence $\{F * \sigma_n\}$ also converges to zero uniformly on compact subsets of Ω as $n \to \infty$.

Theorem 3.3. *If F is a Boehmian of analytic type such that $F = 0$ on some open arc Ω, then $F \equiv 0$.*

Proof. Let $F = f_n/\delta_n \in \beta$ be a Boehmian of analytic type such that $F = 0$ on Ω. Since $\widehat{F}(n) = 0$ for $n = -1, -2, \ldots$, we see that for each n

$$\hat{f}_n(k) = \widehat{F}(k)\hat{\delta}_n(k) = 0 \quad \text{for } k = -1, -2, \ldots. \tag{3.2}$$

Thus, by a well known result (see [5, Theorem 17.18]), f_n is identically zero provided it vanishes on a set of positive measure. Hence to complete the proof it suffices to show that there exist an $n_0 \in \mathbb{N}$ and an arc J such that for each $n \geq n_0$, f_n vanishes on J.

Now,

$$f_n = f_n - (f_n * \delta_k) + (f_n * \delta_k), \quad \text{for all } n, k \in \mathbb{N}. \tag{3.3}$$

Since $\{\delta_k\}$ is a delta sequence, for each n

$$f_n * \delta_k \longrightarrow f_n \text{ uniformly on } T \quad \text{as } k \longrightarrow \infty. \tag{3.4}$$

Let J be any closed subinterval of Ω. Then there exist an $\alpha > 0$ and a closed interval I such that $J \subset I \subset \Omega$ and $(-\alpha, \alpha) + J \subseteq I$. Also, there exists an $n_0 \in \mathbb{N}$ such that $\operatorname{supp} \delta_n \subseteq (-\alpha, \alpha)$, for all $n \geq n_0$.

Now, let $\varepsilon > 0$. Since $f_k \to 0$ uniformly on I as $k \to \infty$, there exists a $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, $|f_k(x)| < \varepsilon$ for all $x \in I$.

Let n be any fixed integer greater than n_0. Then for all $k \geq k_0$

$$\begin{aligned} \left|(f_n * \delta_k)(x)\right| &= \left|(f_k * \delta_n)(x)\right| \\ &\leq \frac{1}{2\pi}\int_{-\alpha}^{\alpha} \left|f_k(x-t)\right| \delta_n(t)\,dt \\ &< \frac{\varepsilon}{2\pi}\int_{-\alpha}^{\alpha} \delta_n(t)\,dt = \varepsilon, \quad \text{for all } x \in J. \end{aligned} \tag{3.5}$$

That is, for each $n \geq n_0$,

$$f_n * \delta_k \longrightarrow 0 \text{ uniformly on } J \quad \text{as } k \longrightarrow \infty, \tag{3.6}$$

By combining (3.3), (3.4), and (3.6), we see that for each $n \geq n_0$, f_n vanishes on J. This establishes the theorem. □

4. Some final comments. In the definition of a delta sequence, the condition that δ_n be nonnegative can be relaxed. In this case, an additional condition must be required. That is, the condition that $1/2\pi \int_{-\pi}^{\pi} |\delta_n(x)| dx \leq M$ for some $M > 0$ and all $n \in \mathbb{N}$ is needed.

A slight modification of the proof of Theorem 3.3 shows that the theorem is still valid if one uses delta sequences in the above sense. The details are left to the reader.

By using this relaxed form of a delta sequence to construct the space of Boehmians, it is not known whether any new Boehmians are obtained.

We conclude this paper with an open problem: What type of function in the unit disk has a Boehmian of analytic type as a boundary value?

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