ABEL-TYPE WEIGHTED MEANS TRANSFORMATIONS INTO $\ell$

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Abstract. Let $q_k = \binom{k+\alpha}{k}$ for $\alpha > -1$ and $Q_n = \sum_{k=0}^{n} q_k$. Suppose $A_q = \{a_{nk}\}$, where $a_{nk} = q_k/Q_n$ for $0 \leq k \leq n$ and 0 otherwise. $A_q$ is called the Abel-type weighted mean matrix. The purpose of this paper is to study these transformations as mappings into $\ell$. A necessary and sufficient condition for $A_q$ to be $\ell$-$\ell$ is proved. Also some other properties of the $A_q$ matrix are investigated.

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1. Introduction. Throughout this paper, we assume that $\alpha > -1$ and $Q_n$ is the partial sums of the sequence $\{q_k\}$, where $q_k$ is as above. Let $A_q = \{a_{nk}\}$. Then the Abel-type weighted mean matrix, denoted by $A_q$, is defined by

$$a_{nk} = \begin{cases} \frac{q_k}{Q_n} & \text{for } 0 \leq k \leq n, \\ 0 & \text{for } k > n. \end{cases}$$

(1.1)

The $A_q$ matrix is the weighted mean matrix that is associated with the Abel-type matrix introduced by M. Lemma in [5]. It is regular, indeed, totally regular.

2. Basic notation and definitions. Let $A = (a_{nk})$ be an infinite matrix defining a sequence summability transformation given by

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k,$$

(2.1)

where $(Ax)_n$ denotes the $n$th term of the image sequence $Ax$. Let $y$ be a complex number sequence. Throughout this paper, we use the following basic notation and definitions:

(i) $c = \{\text{The set of all convergent complex sequences}\}$,

(ii) $\ell = \{y : \sum_{k=0}^{\infty} |y_k| < \infty\}$,

(iii) $\ell^p = \{y : \sum_{k=0}^{\infty} |y_k|^p < \infty\}$,

(iv) $\ell(A) = \{y : Ay \in \ell\}$,

(v) $G = \{y : y_k = O(r^k) \text{ for some } r \in (0,1)\}$,

(vi) $G_w = \{y : y_k = O(r^k) \text{ for some } r \in (0,w), 0 < w < 1\}$.
**Definition 1.** If $X$ and $Y$ are sets of complex number sequences, then the matrix $A$ is called an $X$-$Y$ matrix if the image $Au$ of $u$ under the transformation $A$ is in $Y$, whenever $u$ is in $X$.

**3. Some basic facts.** The following facts are used repeatedly.

1. For any real number $\alpha > -1$ and any nonnegative integer $k$, we have

$$\binom{k + \alpha}{k} \sim \frac{k^\alpha}{\Gamma(\alpha + 1)} \quad \text{(as } k \to \infty\text{)}.$$  

(3.1)

2. For any real number $\alpha > -1$, we have

$$\sum_{k=0}^n \binom{k + \alpha}{k} = \binom{n + \alpha + 1}{n}.$$  

(3.2)

3. Suppose $\{a_n\}$ is a sequence of nonnegative numbers with $a_0 > 0$, that

$$A_n = \sum_{k=0}^n a_k \to \infty.$$  

(3.3)

Let

$$a(x) = \sum_{k=0}^\infty a_k x^k, \quad A(x) = \sum_{k=0}^\infty A_k x^k,$$  

and suppose that

$$a(x) < \infty \quad \text{for } 0 < x < 1.$$  

(3.5)

Then it follows that

$$(1-x)A(x) = a(x) \quad \text{for } 0 < x < 1.$$  

(3.6)

4. The main results

**Lemma 1.** If $A_\ell$ is an $\ell$-$\ell$ matrix, then $1/Q \in \ell$.

**Proof.** By the Knopp-Lorentz theorem [4], $A_\ell$ is an $\ell$-$\ell$ matrix implies that

$$\sum_{k=0}^\infty |a_{n,0}| < \infty,$$  

(4.1)

and consequently we have $1/Q \in \ell$. □

**Lemma 2.** We have that $1/Q \in \ell$ if and only if $\alpha > 0$.

**Proof.** By using (3.1), we have

$$\frac{1}{Q_n} \sim \frac{\Gamma(\alpha + 2)}{n^{\alpha+1}}$$  

(4.2)

and hence the assertion easily follows. □
**Lemma 3.** If $1/Q \in \ell$, then $A_q$ is an $\ell$-$\ell$ matrix.

**Proof.** By Lemma 2, we have $\alpha > 0$. To show that $A_q$ is an $\ell$-$\ell$ matrix, we must show that the condition of the Knopp-Lorentz theorem [4] holds. Using (3.1), we have

$$
\sum_{n=0}^{\infty} |a_{nk}| = \left( \frac{k + \alpha}{k} \right) \sum_{n=k}^{\infty} \frac{1}{Q^n} = \left( \frac{k + \alpha}{k} \right) \sum_{n=k}^{\infty} \left( \frac{1}{Qn} \right)
\leq M_1 K^\alpha \sum_{n=k}^{\infty} \frac{1}{n^{\alpha+1}}
\leq M_1 M_2 k^\alpha \int_{k}^{\infty} \frac{dx}{x^{\alpha+1}}
= M_1 M_2 \frac{\alpha}{\alpha+1}.
$$

(4.3)

Hence, by the Knopp-Lorentz theorem [4], $A_q$ is an $\ell$-$\ell$ matrix.

**Theorem 1.** The following statements are equivalent:

1. $A_q$ is an $\ell$-$\ell$ matrix;
2. $1/Q \in \ell$;
3. $\alpha > 0$.

**Proof.** The theorem easily follows by Lemmas 1, 2, and 3.

**Remark 1.** In Theorem 1, we showed that $A_q$ is an $\ell$-$\ell$ matrix if and only if $1/Q \in \ell$. But the converse is not true in general for any weighted mean matrix $W_p$ that corresponds to a sequence-to-sequence variant of the general $J_p$ power series method of summability [1]. To see this, let

$$
p_k = (\ln(k+2))^\alpha, \quad \alpha > 1.
$$

(4.4)

We show that $1/P \in \ell$ but $W_p$ is not an $\ell$-$\ell$ matrix. We have

$$
P_n = \sum_{k=0}^{n} (\ln(k+2))^\alpha
\sim \int_{0}^{n} (\ln(x+2))^\alpha \, dx \quad \text{(by [6, Thm. 1.20])}
\sim (n+2)(\ln(n+2))^\alpha,
$$

(4.5)

using integration by parts repeatedly. This yields

$$
\frac{1}{P_n} \sim \frac{1}{(n+2)(\ln(n+2))^\alpha}
$$

(4.6)

and by the condensation test, it follows that $1/P \in \ell$. 


Next, we show that $W_p$ is not an $\ell$-$\ell$ matrix by showing that the condition of the Knopp-Lorentz theorem [4] fails to hold. Using (4.6), it follows that
\[
\sum_{n=0}^{\infty} |a_{nk}| = (\ln(k+2))^\alpha \sum_{n=k}^{\infty} \frac{1}{p_n^\alpha} \geq M_1 (\ln(k+2))^{\alpha} \sum_{n=k}^{\infty} \frac{1}{(n+2)(\ln(n+2))^\alpha} \quad \text{for some } M_1 > 0
\]
\[
\geq M_1 M_2 (\ln(k+2))^{\alpha} \int_k^{\infty} \frac{dx}{(x+2)(\ln(x+2))^{\alpha}} \quad \text{for some } M_2 > 0
\]
\[
= \frac{M_1 M_2}{\alpha-1} (\ln(k+2)).
\]
Thus, we have
\[
\sup_k \left\{ \sum_{n=0}^{\infty} a_{nk} \right\} = \infty,
\]
and hence $W_p$ is not an $\ell$-$\ell$ matrix.

**Corollary 1.** $A_q$ is an $\ell$-$\ell$ matrix.

**Proof.** Since $Q_n = \left(\frac{n+\alpha+1}{n}\right)$ and $\alpha > -1$ implies that $\alpha + 1 > 0$, the assertion easily follows by Theorem 1.

**Corollary 2.** $A_q$ is an $\ell$-$\ell$ matrix if and only if $\lim_n (Q_n/nq_n) < 1$.

**Proof.** By Theorem 1, $A_q$ is an $\ell$-$\ell$ matrix implies that $\alpha > 0$, and as a consequence we have $1/(\alpha + 1) < 1$. Now using (3.1), we have
\[
\lim_n \left(\frac{Q_n}{nq_n}\right) = \lim_n \frac{n^{\alpha+1} \Gamma(\alpha+1)}{\Gamma(\alpha+2)n^{\alpha+1}} = \frac{1}{\alpha+1} < 1.
\]
Conversely, if $\lim_n (Q_n/nq_n) < 1$, then it follows from (4.9) that $1/(\alpha + 1) < 1$ and consequently we have $\alpha > 0$, and hence, by Theorem 1, $A_q$ is an $\ell$-$\ell$ matrix.

**Corollary 3.** Suppose that $z_k = \left(\frac{k^\beta}{k}\right)$ and $\alpha < \beta$; then $A_z$ is an $\ell$-$\ell$ matrix whenever $A_q$ is an $\ell$-$\ell$ matrix.

**Proof.** The corollary follows easily by Theorem 1.

**Lemma 4.** If the Abel-type matrix $A_{\alpha,t}$ [5] is an $\ell$-$\ell$ matrix, then $A_{\alpha+1,t}$ is also an $\ell$-$\ell$ matrix.

**Proof.** By the Knopp-Lorentz theorem [4], $A_{\alpha,t}$ is an $\ell$-$\ell$ matrix implies that
\[
\sup_k \left\{ \sum_{n=0}^{\infty} |a_{nk}| \right\} < \infty.
\]
This is equivalent to
\[
\sup_k \left\{ \left(\frac{k+\alpha}{k}\right) \sum_{n=0}^{\infty} t_n^k (1-t_n)^{\alpha+1} \right\} < \infty.
\]
Now from (4.11), we can easily conclude that
\[
\sup_k \left\{ \left( \frac{k + \alpha + 1}{k} \right) \sum_{n=0}^{\infty} t_n^k (1 - t_n)^{\alpha + 2} \right\} < \infty.
\] (4.12)

Hence, \( A_{\alpha+1,t} \) is an \( \ell \)-\( \ell \) matrix.

The next theorem compares the summability fields of the matrices \( A_q \) and \( A_{\alpha,t} \) [5].

**Theorem 2.** If \( A_{\alpha,t} \) and \( A_q \) are \( \ell \)-\( \ell \) matrices, then \( \ell(A_q) \subseteq \ell(A_{\alpha,t}) \).

**Proof.** Let \( x \in \ell(A_q) \). Then we show that \( x \in \ell(A_{\alpha,t}) \). Let \( y \) be the \( A_q \)-transform of the sequence \( x \). Then we have
\[
y_n Q_n = \sum_{k=0}^{n} q_k x_k.
\] (4.13)

Now since \( y_n Q_n \) is the partial sums of the sequence \( q_x \), using (3.6) it follows that
\[
(1 - t_n) \sum_{k=0}^{\infty} Q_k y_k t_n^k = \sum_{k=0}^{\infty} q_k x_k t_n^k.
\] (4.14)

This yields
\[
(1 - t_n)^{\alpha + 2} \sum_{k=0}^{\infty} Q_k y_k t_n^k = (1 - t_n)^{\alpha + 2} \sum_{k=0}^{\infty} q_k x_k t_n^k,
\] (4.15)

and as a consequence we have \( (A_{\alpha+1,t} y)_n = (A_{\alpha,t} x)_n \). By Lemma 4, \( A_{\alpha,t} \) is an \( \ell \)-\( \ell \) matrix implies that \( A_{\alpha+1,t} \) is also an \( \ell \)-\( \ell \) matrix, and from the assumption that \( x \in \ell(A_q) \), it follows that \( y \in \ell \). Consequently, we have \( A_{\alpha+1,t} y \in \ell \) and this is equivalent to \( A_{\alpha,t} x \in \ell \). Thus, \( x \in \ell(A_{\alpha,t}) \) and hence our assertion follows. \( \square \)

**Remark 2.** Theorem 2 gives an important inclusion result in the \( \ell \)-\( \ell \) setting that parallels the famous inclusion result that exists between the power series method of summability and its corresponding weighted mean in the \( c \)-\( c \) setting [1].

**Lemma 5.** Suppose \( A = \{ a_{nk} \} \) is an \( \ell \)-\( \ell \) matrix such that \( a_{nk} = 0 \) for \( k > n, m > s \) (both positive integers); then \( \ell(A^s) \subseteq \ell(A^m) \), where the interpretation for \( A^s \) and \( A^m \) is as given in [6, p. 28].

**Theorem 3.** If \( B = A_q \) is an \( \ell \)-\( \ell \) matrix, then \( B^m \) is also an \( \ell \)-\( \ell \) matrix (for \( m \) a positive integer greater than 1.)

**Proof.** Let \( x \in \ell \), \( B \) is an \( \ell \)-\( \ell \) matrix implies that \( x \in \ell(B) \). By Lemma 5, we have \( \ell(B) \subseteq \ell(B^m) \) and hence it follows that \( x \in \ell(B^m) \). Hence, \( B^m \) is an \( \ell \)-\( \ell \) matrix. \( \square \)

**Remark 3.** Theorem 3 gives a result that goes parallel to a \( c \)-\( c \) result given on [6, Thm. 2.4, p. 28].

In Corollary 1, we showed that \( A_Q \) is an \( \ell \)-\( \ell \) matrix. Here, a question may be raised as to whether \( A_Q \) maps \( \ell^p \) into \( \ell \) for \( p > 1 \). But this is answered negatively by the following theorem.
Theorem 4. $A_Q$ does not map $\ell^p$ into $\ell$ for $p > 1$.

Proof. Let $A_Q = \{b_{nk}\}$. Note that if $A_{Q,\alpha}$ maps $\ell^p$ into $\ell$, then by [3, Thm. 2], we must have

$$\lim_k \sum_{n=1}^\infty |b_{nk}| = 0. \quad (4.16)$$

Let

$$R_n = \sum_{k=1}^n Q_k, \quad (4.17)$$

then it follows that

$$\sum_{n=1}^\infty b_{nk} = \binom{k + \alpha + 1}{k} \sum_{n=1}^\infty \frac{1}{R_n} = \binom{k + \alpha + 1}{k} \sum_{n=k}^\infty \frac{1}{n^{\frac{\alpha + 2}{k}}} \geq M_1 k^{\alpha+1} \sum_{n=k}^\infty \frac{1}{n^{\alpha+2}} \quad \text{for some } M_1 > 0 \quad (4.18)$$

$$\geq M_1 M_2 k^{\alpha+1} \int_k^\infty \frac{dx}{x^{\alpha+2}} \quad \text{for some } M_2 > 0$$

Thus, it follows that

$$\lim_k \sum_{n=1}^\infty |b_{nk}| > 0, \quad (4.19)$$

and hence $A_Q$ does not map $\ell^p$ into $\ell$ for $p > 1$ by [3, Thm. 2].

Our next theorem has the form of an extension mapping theorem. It indicates that a mapping of $A_q$ from $G$ or $G_w$ into $\ell$ can be extended to a mapping of $\ell$ into $\ell$.

Theorem 5. The following statements are equivalent:

1. $A_q$ is an $\ell$-$\ell$ matrix;
2. $A_q$ is a $G$-$\ell$ matrix;
3. $A_q$ is a $G_w$-$\ell$ matrix.

Proof. Since $G$ is a subset of $\ell$ and $G_w$ a subset of $G$, (1)$\Rightarrow$(2)$\Rightarrow$(3) follow easily. The assertion that (3)$\Rightarrow$(1) follows by [7, Thm. 1.1] and Theorem 1.

Corollary 4. (1) $A_Q$ is a $G$-$\ell$ matrix.
(2) $A_Q$ a $G_w$-$\ell$ matrix.

Proof. Since $A_Q$ is an $\ell$-$\ell$ matrix by Corollary 1, the assertion follows by Theorem 5.

Corollary 5. (1) If $A_q$ is a $G$-$G$ matrix, then $A_q$ is an $\ell$-$\ell$ matrix.
(2) If $A_q$ is a $G_w$-$G_w$ matrix, then $A_q$ is an $\ell$-$\ell$ matrix.

Proof. The assertion follows easily by Theorem 5.
**THEOREM 6.** \( A_q \) is a \( G-G \) matrix if and only if \( 1/Q \in G \).

**Proof.** If \( A_q \) is a \( G-G \) matrix, then the first column of \( A_q \) is must in \( G \). This gives \( 1/Q \in G \) since \( a_{n,0} = q_0/Q_n \). Conversely, suppose \( 1/Q \in G \). Then \( 1/Q_n \leq M_1 r^n \) for \( M_1 > 0 \) and \( r \in (0,1) \). Now let \( u \in G \), say \( |u_k| \leq M_2 t^k \) for some \( M_2 > 0 \) and \( t \in (0,1) \). Let \( Y \) be the \( A_q \)-transform of the sequence \( u \). Then we have

\[
|Y_n| \leq M_1 M_2 r^n \sum_{k=0}^{n} \binom{k + \alpha}{k} t^k < M_1 M_2 r^n (1 - t)^{-(\alpha+1)} < M_3 r^n \quad \text{for some} M_3 > 0.
\]

(4.20)

Therefore, \( Y \in G \) and hence it follows that \( A_q \) is a \( G-G \) matrix.

**THEOREM 7.** \( A_q \) is a \( G_w-G_w \) matrix if and only if \( 1/Q \in G_w \).

**Proof.** The proof follows easily using the same steps as in the proof of Theorem 6 by replacing \( G \) with \( G_w \).

**LEMMA 6.** If the Abel-type matrix \( A_{\alpha,t} \) [5] is a \( G-G \) matrix, then \( A_{\alpha+1,t} \) is also a \( G-G \) matrix.

**Proof.** By [5, Thm. 7], \( A_{\alpha,t} \) is \( G-G \) implies that \( (1 - t)^{\alpha+1} \in G \). But \( (1 - t)^{\alpha+1} \in G \) yields \( (1 - t)^{\alpha+2} \in G \), and hence by [5, Thm. 7], it follows that \( A_{\alpha+1,t} \) is a \( G-G \) matrix.

**LEMMA 7.** If the Abel-type matrix \( A_{\alpha,t} \) [5] is a \( G_w-G_w \) matrix, then \( A_{\alpha+1,t} \) is also a \( G_w-G_w \) matrix.

**Proof.** The assertion easily follows by replacing \( G \) with \( G_w \) in the proof of Lemma 6.

**THEOREM 8.** If \( A_{\alpha,t} \) [5] and \( A_q \) are \( G-G \) matrices, then the \( G(A_{\alpha,t}) \) contains \( G(A_q) \).

**Proof.** The proof easily follows using the same techniques as in the proof of Theorem 3 by replacing \( \ell \) with \( G \) and applying Lemma 6.

**THEOREM 9.** If \( A_{\alpha,t} \) [3] and \( A_q \) are \( G_w-G_w \) matrices, then \( G_w(A_{\alpha,t}) \) contains \( G_w(A_q) \).

**Proof.** The proof easily follows using the same techniques as in the proof of Theorem 3 by replacing \( \ell \) with \( G_w \) and applying Lemma 7.

**References**


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