LEFT GLOBAL DIMENSIONS AND INVERSE POLYNOMIAL MODULES

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(Received 20 November 1999)

ABSTRACT. We prove the fact $\text{l.gl.dim } R[x] = (\text{l.gl.dim } R) + 1$, where l.gl.dim means the left global dimension by using inverse polynomial modules and injective dimensions. The classical way to prove the fact $\text{l.gl.dim } R[x] = (\text{l.gl.dim } R) + 1$ is using polynomial modules and projective dimensions.

Keywords and phrases. Inverse polynomial module, injective module, short exact sequence, left global dimension.

2000 Mathematics Subject Classification. Primary 16E30, 13C11, 16D80.

1. Introduction. The classical way to prove the fact $\text{l.gl.dim } R[x] = (\text{l.gl.dim } R) + 1$, where l.gl.dim means the left global dimension is using the construction $M[x]$ (where M is any left R -module). In this paper, we give another proof of this fact by using inverse polynomial module $M[x^{-1}]$ and injective dimensions instead of polynomial module $M[x]$ and projective dimensions. Northcott [3] and McKerrow [1] showed that the polynomial module $M[x]$ and the inverse polynomial module $M[x^{-1}]$ are not isomorphic as left $R[x]$ -modules by showing that if R is a left noetherian ring and $E \neq 0$ is an injective left R -module, then $E[x^{-1}]$ is an injective left $R[x]$ -module while $E[x]$ is not an injective left $R[x]$ -module. Park in [5] also showed that if $P \neq 0$ is a projective left R -module, then $P[x^{-1}]$ is not a projective left $R[x]$ -module while $P[x]$ is a projective left $R[x]$ -module. Inverse polynomial modules were developed in [1, 3, 4, 5], and recently in [2].

DEFINITION 1.1. Let R be a ring and M be a left R -module, then $M[x^{-1}]$ is a left $R[x]$ -module such that

$$x(m_0 + m_1x^{-1} + \cdots + m_nx^{-n}) = m_1 + m_2x^{-1} + \cdots + m_nx^{-n+1} \quad (1.1)$$

and such that

$$r(m_0 + m_1x^{-1} + \cdots + m_nx^{-n}) = rm_0 + rm_1x^{-1} + \cdots + rm_nx^{-n}, \quad (1.2)$$

where $r \in R$. Similarly, we can define $M[[x^{-1}]]$ as a left $R[x]$ -module.

LEMMA 1.2. *If $E[[x^{-1}]]$ is an injective left $R[x]$ -module, then E is an injective left R -module.*

PROOF. Let $I \subset R$ be a left ideal and let $f: I \rightarrow E$ be a R -linear map. Then the map $I[[x^{-1}]] \rightarrow E[[x^{-1}]] (\sum_{i=0}^{\infty} r_i x^{-i} \rightarrow \sum_{i=0}^{\infty} f(r_i) x^{-i})$ is an $R[x]$ -linear map so can be

extended to a map

$$R[[x^{-1}]] \xrightarrow{g} E[[x^{-1}]]. \quad (1.3)$$

Since $xR = 0$ (for $R \subset R[[x^{-1}]]$), $xg(R) = 0$ in $E[[x^{-1}]]$. But this implies $g(R) \subset E$ (with $E \subset E[[x^{-1}]]$). Then the map $R \rightarrow E$ agreeing with g is an R -linear map and extends $I \rightarrow E$. So E is an injective R -module. $\square$

THEOREM 1.3. *Let M be a left R -module, then*

$$\text{inj dim}_{R[x]}(M[[x^{-1}]]) = \text{inj dim}_R(M). \quad (1.4)$$

PROOF. Let R be a ring and E be an injective left R -module. Define $\phi : \text{Hom}_R(R[x], E) \rightarrow E[[x^{-1}]]$ by $\phi(f) = f(1) + f(x)x^{-1} + f(x^2)x^{-2} + \dots$, then $\text{Hom}_R(R[x], E)$ and $E[[x^{-1}]]$ are isomorphic as left $R[x]$ -modules. Since $\text{Hom}_R(R[x], E)$ is an injective left $R[x]$ -module so is $E[[x^{-1}]]$. Suppose $\text{inj dim}_R(M) = n$ and

$$0 \rightarrow M \rightarrow E^0 \rightarrow E^1 \rightarrow \dots \rightarrow E^n \rightarrow 0 \quad (1.5)$$

is an injective resolution of M . Then

$$0 \rightarrow M[[x^{-1}]] \rightarrow E^0[[x^{-1}]] \rightarrow E^1[[x^{-1}]] \rightarrow \dots \rightarrow E^n[[x^{-1}]] \rightarrow 0 \quad (1.6)$$

is an injective resolution of $M[[x^{-1}]]$. Let

$$K^i = \ker(E^i \rightarrow E^{i+1}) \quad \text{for } 0 \leq i < n. \quad (1.7)$$

Then K^i is not an injective R -module for $0 \leq i < n$. So by the above lemma $K^i[[x^{-1}]]$ is not an injective $R[x]$ -module. So then we get

$$\text{inj dim}_{R[x]}(M[[x^{-1}]]) = n. \quad (1.8)$$

Suppose $\text{inj dim}_R(M) = \infty$ and

$$0 \rightarrow M \rightarrow E^0 \rightarrow E^1 \rightarrow \dots \rightarrow E^n \rightarrow \dots \quad (1.9)$$

is an injective resolution of M . Then

$$0 \rightarrow M[[x^{-1}]] \rightarrow E^0[[x^{-1}]] \rightarrow E^1[[x^{-1}]] \rightarrow \dots \rightarrow E^n[[x^{-1}]] \rightarrow \dots \quad (1.10)$$

is an injective resolution of $M[[x^{-1}]]$. But K^i is not an injective R -module for all i . Thus $K^i[[x^{-1}]]$ is not an injective $R[x]$ -module for all i . Therefore, $\text{inj dim}_{R[x]}(M[[x^{-1}]]) = \infty$. Similarly, if

$$\text{inj dim}_{R[x]}(M[[x^{-1}]]) = n, \quad \text{then } \text{inj dim}_R(M) = n \quad (1.11)$$

and if

$$\text{inj dim}_{R[x]}(M[[x^{-1}]]) = \infty, \quad \text{then } \text{inj dim}_R(M) = \infty. \quad (1.12)$$

Hence, $\text{inj dim}_{R[x]}(M[[x^{-1}]]) = \text{inj dim}_R(M)$. $\square$

THEOREM 1.4. *Let $M, M[[X^{-1}]]$ be left $R[x]$ -modules. Then there is a short exact sequence of $R[x]$ -modules*

$$0 \longrightarrow M \longrightarrow M[[x^{-1}]] \longrightarrow M[[x^{-1}]] \longrightarrow 0. \quad (1.13)$$

PROOF. Let $\phi : M \rightarrow [[x^{-1}]]$ be defined by

$$\phi(y) = y + (xy)x^{-1} + (x^2y)x^{-2} + (x^3y)x^{-3} + \cdots, \quad \text{for } y \in M, \quad (1.14)$$

then ϕ is an injective $R[x]$ -linear map. Let $\psi : M[[x^{-1}]] \rightarrow M[[x^{-1}]]$ be defined by

$$\psi(m_0 + m_1x^{-1} + m_2x^{-2} + \cdots) = (m_1 - xm_0) + (m_2 - xm_1)x^{-1} + \cdots, \quad (1.15)$$

then ψ is a surjective $R[x]$ -linear map. Let y be an element of M , then

$$\begin{aligned} (\psi \circ \phi)(y) &= \psi(y + (xy)x^{-1} + (x^2y)x^{-2} + \cdots) \\ &= (xy - xm_0) + (x^2y - x^2y)x^{-1} + (x^3y - x^3y)x^{-2} + \cdots = 0. \end{aligned} \quad (1.16)$$

Therefore, $\text{image}(\phi) \subset \ker(\psi)$. Let $m_0 + m_1x^{-1} + m_2x^{-2} + \cdots$ be an element of $\ker(\psi)$, then

$$\begin{aligned} \psi(m_0 + m_1x^{-1} + m_2x^{-2} + \cdots) \\ = (m_1 - xm_0) + (m_2 - xm_1)x^{-1} + (m_3 - xm_2)x^{-2} + \cdots = 0. \end{aligned} \quad (1.17)$$

Therefore, $m_{i+1} = xm_i$, for all $i = 0, 1, 2, \dots$. Then, for $m_0 \in M$,

$$\begin{aligned} \phi(m_0) &= m_0 + (xm_0)x^{-1} + (x^2m_0)x^{-2} + \cdots \\ &= m_0 + m_1x^{-1} + m_2x^{-2} + \cdots, \quad \text{since } m_{i+1} = xm_i. \end{aligned} \quad (1.18)$$

So, $m_0 + m_1x^{-1} + \cdots \in \text{image}(\phi)$. Thus, $\ker(\psi) \subset \text{image}(\phi)$. Therefore, $\text{image}(\phi) = \ker(\psi)$. Hence, $0 \rightarrow M \rightarrow M[[x^{-1}]] \rightarrow M[[x^{-1}]] \rightarrow 0$ is a short exact sequence of $R[x]$ -modules. $\square$

COROLLARY 1.5. $\text{l.gl.dim } R \leq \text{l.gl.dim } R[x] \leq \text{l.gl.dim } R + 1$.

PROOF. The first inequality follows from Theorem 1.3. The second follows from Theorems 1.4 and 1.3. $\square$

2. Main theorem

THEOREM 2.1. $\text{l.gl.dim } R[x] = (\text{l.gl.dim } R) + 1$.

PROOF. From Corollary 1.5 we see that we only need argue that if $M \neq 0$ is a left R -module and if $\text{inj dim } M = n < \infty$, then there is a left $R[x]$ -module N with $\text{inj dim } N = n + 1$. In fact we show that M itself can be made into such an $R[x]$ -module. Let $M \neq 0$ be a left R -module and let $\text{inj dim } M = n < \infty$, and make M into an $R[x]$ -module with $xM = 0$. By induction on n . Consider the short exact sequence of $R[x]$ -modules

$$0 \longrightarrow M \longrightarrow M[[x^{-1}]] \longrightarrow M[[x^{-1}]] \longrightarrow 0. \quad (2.1)$$

From this short exact sequence we see that $\text{injdim}_{R[x]} M \leq n + 1$, since $\text{injdim}_{R[x]} M[[x^{-1}]] = \text{injdim}_R M = n$. So we only need to prove that for any n , if $\text{injdim}_R M = n$, then $\text{injdim}_{R[x]} M > n$. If $n = 0$, then M is an injective left R -module. But any injective $R[x]$ -module is x -divisible. Also $xM = 0$. So M is not x -divisible. Hence, M is not an injective $R[x]$ -module. Now suppose $n = 1$. Then we have an exact sequence

$$0 \rightarrow M \rightarrow E \rightarrow \frac{E}{M} \rightarrow 0 \quad (2.2)$$

with E and E/M injective left R -modules. Since M is a submodule of $E[[x^{-1}]]$ which is an injective left $R[x]$ -module as left $R[x]$ -module, we have the short exact sequence of $R[x]$ -modules

$$0 \rightarrow M \rightarrow E[[x^{-1}]] \rightarrow \frac{E[[x^{-1}]]}{M} \rightarrow 0. \quad (2.3)$$

Now we want to argue $E[[x^{-1}]]/M$ is not an injective left $R[x]$ -module. Suppose $E[[x^{-1}]]/M$ is an injective left $R[x]$ -module. Let $I = (x) \subset R[x]$. Consider the submodule of all z in $E[[x^{-1}]]/M$ such that $xz = 0$. Then this submodule of $E[[x^{-1}]]/M$ is isomorphic to $E/M \oplus Mx^{-1}$ as an $R[x]/(x) \cong R$ module, i.e., is isomorphic to $E/M \oplus M$. So if $E[[x^{-1}]]/M$ were an injective left $R[x]$ -module, then $E/M \oplus M$, and so M , would be an injective left R -module. Therefore by this contradiction we see that $E[[x^{-1}]]/M$ is not an injective left $R[x]$ -module. Now we suppose $\text{injdim}_R M = n > 1$ and make the induction hypothesis. Let

$$0 \rightarrow M \rightarrow E \rightarrow C \rightarrow 0 \quad (2.4)$$

be an exact sequence of left R -modules with E an injective left R -module. Then $\text{injdim}_R C = n - 1$. Make this into an exact sequence of left $R[x]$ -modules with $xM = 0$, $xE = 0$, and $xC = 0$. Then by the induction hypothesis $\text{injdim}_{R[x]} E = 1$, $\text{injdim}_{R[x]} C = (n - 1) + 1 = n$. Then since $n > 1$ we get that $\text{injdim}_{R[x]} M = n + 1$. This implies $\text{l.gl.dim } R[x] = (\text{l.gl.dim } R) + 1$. $\square$

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