ON THE REPRESENTATION OF m AS $\sum_{k=-n}^n \epsilon_k k$

LANE CLARK

(Received 3 September 1998)

ABSTRACT. Let $A(n, m)$ be the number of solutions of $\sum_{k=-n}^n \epsilon_k k = m$ where each $\epsilon_k \in \{0, 1\}$. We determine the asymptotic behavior of $A(n, m)$ for $m = o(n^{3/2})$, extending results of van Lint and of Entringer.

Keywords and phrases. Representation, asymptotic behavior.

2000 Mathematics Subject Classification. 11A67, 11B75.

For a nonnegative integer n and an integer m , let

$$A(n, m) = \# \left\{ (\epsilon_{-n}, \dots, \epsilon_0, \dots, \epsilon_n) \in \{0, 1\}^{2n+1} : \sum_{k=-n}^n \epsilon_k k = m \right\}. \quad (1)$$

van Lint [2] answered a question of Erdős by determining the asymptotic behavior of $A(n, 0)$. Entringer [1] used this result and induction to determine the asymptotic behavior of $A(n, m)$ for $m = O(n)$. In this note, we give a further extension by showing that

$$A(n, m) \sim \left(\frac{3}{\pi} \right)^{1/2} 2^{2n+1} n^{-3/2} \quad \text{as } n \rightarrow \infty, \quad (2)$$

for $m = o(n^{3/2})$. We estimate the integral below, as in [2], though our analysis is more involved. It is immediately seen that $A(n, m)$ is the coefficient of z^m in $\prod_{k=-n}^n (1 + z^k)$ and, hence,

$$\begin{aligned} A(n, m) &= \frac{1}{2\pi i} \oint_C \frac{\prod_{k=-n}^n (1 + z^k)}{z^{m+1}} dz \\ &= \frac{2^{2n+2}}{\pi} \int_0^{\pi/2} \cos 2mx \prod_{k=1}^n \cos^2 kx \, dx, \end{aligned} \quad (3)$$

upon parameterizing the unit circle C (see [1, 2]). Note that $A(n, m) = A(n, -m)$ and $A(n, m) = 0$ if and only if $|m| > \binom{n+1}{2}$. Hence, we assume that m is a nonnegative integer. We denote the nonnegative integers by $\mathbb{N}$; the integers by $\mathbb{Z}$; and the real numbers by $\mathbb{R}$.

We use the following Taylor series approximations which are valid for all $x \in \mathbb{R}$.

$$\sin x = x - \frac{x^3}{6} + r(x); \quad |r(x)| \leq \frac{x^4}{24} \quad \text{for } x \in \mathbb{R} \quad \text{and} \quad r(x) \geq 0 \quad \text{for } x \in [0, \pi]; \quad (4)$$

$$\cos x = 1 + s(x); \quad |s(x)| \leq |x| \quad \text{for } x \in \mathbb{R}; \quad (5)$$

$$\cos^2 x = 1 - x^2 + t(x); \quad |t(x)| \leq \frac{2|x|^3}{3} \quad \text{for } x \in \mathbb{R}; \quad (6)$$

$$e^{-x} = 1 - x + u(x); \quad 0 \leq u(x) \leq \frac{x^2}{2} \quad \text{for } x \in [0, \infty). \quad (7)$$

Of course, r , s , t , and u are all infinitely-differentiable functions on $\mathbb{R}$. We also use the following standard inequalities:

$$e^{x-x^2} \leq 1 + x \leq e^{x-x^2/6} \quad \text{for } x \in [-0.68, 0.68]; \quad (8)$$

$$1 - x \leq e^{-x} \quad \text{for } x \in \mathbb{R}. \quad (9)$$

For all $n \in \mathbb{Z}$ and $x \in \mathbb{R}$ with $\sin x \neq 0$, (4) gives

$$\frac{\sin nx}{\sin x} = n - \frac{n^3 - n}{6}x^2 + v(n, x), \quad (10)$$

where

$$v(n, x) = \frac{-((n^3 - n)/36)x^5 + ((n^3 - n)/6)x^2r(x) + r(nx) - nr(x)}{x - (x^3/6) + r(x)}, \quad (11)$$

so that

$$|v(n, x)| \leq \frac{n^4x^4/23}{5x/6} = \frac{6}{115}n^4x^3 \quad \text{for } x \in [0, 1] \text{ and } n \geq 20. \quad (12)$$

(Naturally, we define $\sin nx / \sin x = n$ when $x = 0$ to remove that discontinuity.) We require the following result (see [2] for a statement of a version of (a)).

LEMMA. (a) For $(\pi/2n) \leq x \leq \pi/2$ and $n \geq 4$,

$$\left| \frac{\sin nx}{\sin x} \right| \leq \frac{2n}{3}. \quad (13)$$

(b) For $0 \leq x \leq (\pi/2n)$ and $n \geq 20$,

$$\left| \frac{\sin nx}{\sin x} \right| \leq n - \frac{n^3x^2}{12}. \quad (14)$$

PROOF. (a) First, (4) gives $\sin(\pi/2n) \geq (\pi/2n) - (\pi^3/48n^3) \geq (3/2n)$ for $n \geq 4$. Hence,

$$\left| \frac{\sin nx}{\sin x} \right| = \frac{|\sin nx|}{\sin x} \leq \frac{1}{\sin(\pi/2n)} \leq \frac{2n}{3}. \quad (15)$$

(b) Next, (10) gives $n - ((n^3 - n)/6)x^2 + v(n, x) \leq n - n^3x^2((1/6) - (1/6n^2) - (6/115)nx) \leq n - (n^3x^2/12)$ for $n \geq 20$. Hence,

$$\left| \frac{\sin nx}{\sin x} \right| = \frac{\sin nx}{\sin x} \leq n - \frac{n^3x^2}{12}. \quad (16)$$

□

For all $x \in \mathbb{R}$ and $n \geq 1$, (9) gives (see [2])

$$\begin{aligned}
0 &\leq \prod_{k=1}^n \cos^2 kx = \prod_{k=1}^n (1 - \sin^2 kx) \leq \exp \left(- \sum_{k=1}^n \sin^2 kx \right) \\
&= \exp \left(- \frac{n}{2} + \frac{\sin nx \cos(n+1)x}{2 \sin x} \right) \leq \exp \left(- \frac{n}{2} + \frac{1}{2} \left| \frac{\sin nx}{\sin x} \right| \right).
\end{aligned} \tag{17}$$

Hence, for all $m \in \mathbb{N}$ and $n \geq 20$, the lemma and (17) now give

$$\left| \int_{\pi/2n}^{\pi/2} \cos 2mx \prod_{k=1}^n \cos^2 kx \, dx \right| \leq 2e^{-n/6}, \tag{18}$$

and, for all $0 \leq c \leq n^{1/2}$,

$$\left| \int_{cn^{-3/2}}^{\pi/2n} \cos 2mx \prod_{k=1}^n \cos^2 kx \, dx \right| \leq \int_{cn^{-3/2}}^{\pi/2n} e^{-n^3 x^2 / 24} \, dx \leq e^{-c^2 / 24}. \tag{19}$$

If $k \in \mathbb{Z}$ and $x \in \mathbb{R}$, (6) and (7) give

$$\cos^2 kx = e^{-k^2 x^2} (1 + w(k, x)), \tag{20}$$

where

$$w(k, x) = e^{k^2 x^2} (t(kx) - u(k^2 x^2)) \tag{21}$$

is infinitely-differentiable on $\mathbb{R}$ for each integer k and, for $1 \leq k \leq n$, $0 \leq x \leq n^{-1}$,

$$|w(k, x)| \leq 4k^3 x^3. \tag{22}$$

Now, for $0 \leq x \leq an^{-1} \leq 0.5n^{-1}$ and $n \geq 7$,

$$\sum_{k=1}^n (|w(k, x)| + |w(k, x)|^2) \leq 6x^3 \sum_{k=1}^n k^3 \leq 2n^4 x^3 \leq 2a^3 n, \tag{23}$$

so that (8) gives

$$e^{-2a^3 n} \leq \prod_{k=1}^n (1 + w(k, x)) \leq e^{2a^3 n}. \tag{24}$$

Hence, for all $m \in \mathbb{N}$, $0 \leq b \leq 0.5n^{1/2}$, $n \geq 7$, (20) and (24) give with $\sigma = \sigma(n) = n(n+1)(2n+1)/6$,

$$\left| \int_0^{bn^{-3/2}} \cos 2mx \prod_{k=1}^n \cos^2 kx \, dx \right| \leq \int_0^{bn^{-3/2}} e^{-\sigma x^2 + 2b^3 n^{-1/2}} \, dx \leq \frac{be^{2b^3 n^{-1/2}}}{n^{3/2}}. \tag{25}$$

For $0 \leq bn^{-3/2} \leq cn^{-3/2} \leq 0.5n^{-1}$, $n \geq 7$, $t \in \mathbb{Z}$, (20) and (24) give

$$\begin{aligned}
e^{-2c^3 n^{-1/2}} \int_{bn^{-3/2}}^{cn^{-3/2}} x^t e^{-\sigma x^2} \, dx &\leq \int_{bn^{-3/2}}^{cn^{-3/2}} x^t \prod_{k=1}^n \cos^2 kx \, dx \\
&\leq e^{2c^3 n^{-1/2}} \int_{bn^{-3/2}}^{cn^{-3/2}} x^t e^{-\sigma x^2} \, dx.
\end{aligned} \tag{26}$$

Hence,

$$\int_{bn^{-3/2}}^{cn^{-3/2}} \prod_{k=1}^n \cos^2 kx \, dx \sim \frac{(3\pi)^{1/2}}{2} n^{-3/2}, \tag{27}$$

and, for all $m \in \mathbb{N}$,

$$\int_{bn^{-3/2}}^{cn^{-3/2}} s(2mx) \prod_{k=1}^n \cos^2 kx \, dx = O(mn^{-3}), \quad (28)$$

since

$$\int_{bn^{-3/2}}^{cn^{-3/2}} e^{-\sigma x^2} \, dx \sim \frac{(3\pi)^{1/2}}{2} n^{-3/2}, \quad (29)$$

$$\int_{bn^{-3/2}}^{cn^{-3/2}} x e^{-\sigma x^2} \, dx \sim \frac{3}{2} n^{-3}, \quad (30)$$

and (26) holds for all sufficiently large n provided $b = b(n) \rightarrow 0$, $c = c(n) \rightarrow \infty$ with $c = o(n^{1/6})$ as $n \rightarrow \infty$.

Consequently, (5), (18), (19), (25), (27), and (28) give

$$\begin{aligned} \int_0^{\pi/2} \cos 2mx \prod_{k=1}^n \cos^2 kx \, dx &= \int_{(\ln n)^{-1/2} n^{-3/2}}^{7(\ln n)^{1/2} n^{-3/2}} \prod_{k=1}^n \cos^2 kx \, dx \\ &\quad + \int_{(\ln n)^{-1/2} n^{-3/2}}^{7(\ln n)^{1/2} n^{-3/2}} s(2mx) \prod_{k=1}^n \cos^2 kx \, dx \\ &\quad + O((\ln n)^{-1/2} n^{-3/2}) \\ &\sim \frac{(3\pi)^{1/2}}{2} n^{-3/2} \quad \text{as } n \rightarrow \infty, \end{aligned} \quad (31)$$

for all $m = m(n) = o(n^{3/2})$ (our error term being adequate for our analysis which indicates where the integral is concentrated). Hence, (3) gives

$$A(n, m) \sim \left(\frac{3}{\pi}\right)^{1/2} 2^{2n+1} n^{-3/2} \quad \text{as } n \rightarrow \infty, \quad (32)$$

for all $m = m(n) = o(n^{3/2})$. This completes the proof. $\square$

REFERENCES

- [1] R. C. Entringer, *Representation of m as $\sum_{k=-n}^n \epsilon_k k$* , Canad. Math. Bull. **11** (1968), 289–293. MR 37#5138. Zbl 164.05003.
- [2] J. H. van Lint, *Representation of 0 as $\sum_{k=-N}^N \epsilon_k k$* , Proc. Amer. Math. Soc. **18** (1967), 182–184. MR 34#5789. Zbl 152.03401.

CLARK: DEPARTMENT OF MATHEMATICS, SOUTHERN ILLINOIS UNIVERSITY AT CARBONDALE, CARBONDALE, IL 62901-4408, USA