A CHARACTERIZATION OF MÖBIUS TRANSFORMATIONS

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ABSTRACT. We give a new invariant characteristic property of Möbius transformations. Keywords and phrases. Möbius transformations, Schwarzian derivative, Newton derivative. 2000 Mathematics Subject Classification. Primary 30C35.

1. Introduction. Throughout this paper, we let \( w = f(z) \) be a nonconstant meromorphic function in \( \mathbb{C} \) unless otherwise stated.

We consider the following properties.

**Property 1.1.** \( w = f(z) \) transforms circles in the \( z \)-plane onto circles in the \( w \)-plane, including straight lines among circles.

**Property 1.2.** Suppose that \( w = f(z) \) is analytic and univalent in a nonempty simply connected domain \( \mathbb{R} \) on the \( z \)-plane. Let \( ABCD \) be an arbitrary quadrilateral (not self-intersecting) contained in \( \mathbb{R} \). If we set \( A' = f(A), B' = f(B), C' = f(C), D' = f(D) \) and if \( A'B'C'D' \) is a quadrilateral on the \( w \)-plane which is not self-intersecting, then the following hold

\[
\angle A + \angle C = \angle A' + \angle C', \quad \angle B + \angle D = \angle B' + \angle D'.
\]

The following is a well-known principle of circle transformation of Möbius transformations.

**Theorem 1.3.** \( w = f(z) \) satisfies Property 1.1 if and only if \( w = f(z) \) is a Möbius transformation.

In [1], it is shown that Property 1.1 implies Property 1.2 and a new invariant characteristic property of Möbius transformations is given as follows.

**Theorem 1.4.** Let \( \alpha \) be an arbitrary fixed real number such that \( 0 < \alpha < 2\pi \). Suppose that \( w = f(z) \) is analytic and univalent in a nonempty simply connected domain \( \mathbb{R} \) on the \( z \)-plane. Let \( ABCD \) be an arbitrary quadrilateral (not self-intersecting) contained in \( \mathbb{R} \) satisfying

\[
\angle A + \angle C = \alpha.
\]

If \( A' = f(A), B' = f(B), C' = f(C), D' = f(D) \) is a quadrilateral on the \( w \)-plane which is not self-intersecting, then the only function which satisfies

\[
\angle A' + \angle C' = \alpha
\]

is a Möbius transformation.
Theorem 1.4 gives an alternative proof of “the only if part” of Theorem 1.3. Motivated by the above results, we consider the following property.

**PROPERTY 1.5.** Let $k$ be an arbitrary positive real number. For three arbitrary distinct points $a$, $b$, and $c$ in $\mathbb{R}$ satisfying

$$\frac{|a - b|}{|c - b|} = k,$$  \hfill (1.4)

we have

$$\frac{|f(a) - f(b)|}{|f(c) - f(b)|} \cdot \frac{f(c)}{f(a)} = k.$$  \hfill (1.5)

In Section 3, we prove the following result concerning the mapping property of an analytic and univalent function on a connected domain.

**THEOREM 1.6.** Let $k$ be an arbitrary positive real number. Let $w = f(z)$ be analytic and univalent in a nonempty connected domain $\mathbb{D}$ on the $z$-plane such that $f(z) \neq 0$ for all $z \in \mathbb{D}$. Then $f$ satisfies Property 1.5 if and only if $f$ is a Möbius transformation of the form $u/(z + v)$, $u \neq 0$.

2. Lemmas

**DEFINITION 2.1.** Let $f$ be a complex-valued function. The Schwarzian derivative of $f$ is defined as follows:

$$S_f(z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2} \left( \frac{f''(z)}{f'(z)} \right)^2.$$  \hfill (2.1)

Similar to Schwarzian derivative, we have the following.

**DEFINITION 2.2.** Let $f$ be a complex-valued function. We define the Newton derivative of $f$ as follows:

$$N_f(z) = \left( z - \frac{f(z)}{f'(z)} \right)' = \frac{f(z)f''(z)}{(f'(z))^2}.$$  \hfill (2.2)

**REMARK 2.3.** Note that $N_f(z)$ is the first derivative of Newton’s method of $f$.

**REMARK 2.4.** Let $f$ be a complex-valued function. It is well known that $S_f(z) = 0$ if and only if $f$ is a Möbius transformation.

From Remark 2.4, we have observed that a similar result holds true when we replace Schwarzian derivative by the Newton derivative.

**LEMMA 2.5.** Let $f$ be a complex-valued function. Then $N_f(z) = 2$ if and only if $f$ is a Möbius transformation of the form $u/(z + v)$, $u \neq 0$.

**PROOF.** Let $f$ be a Möbius transformation of the form $u/(z + v)$, $u \neq 0$, then it is easily checked that $N_f(z) = 2$. Let $f$ be a complex-valued function such that $N_f(z) = 2$. It follows that

$$\left( z - \frac{f(z)}{f'(z)} \right)' = 2$$  \hfill (2.3)
which implies that
\[
z - \frac{f(z)}{f'(z)} = 2z - c_1, \tag{2.4}
\]
where \(c_1\) is a complex constant, thus
\[
\frac{f(z)}{f'(z)} = -z + c_1 \tag{2.5}
\]
or
\[
\frac{1}{f(z)} \frac{df(z)}{dz} = \frac{1}{-z + c_1}. \tag{2.6}
\]
From which it follows by a simple calculation that \(f\) is a Möbius transformation of the form \(u/(z + v), u \neq 0\).

3. Main result. In this section, we assume that \(w = f(z)\) is analytic and univalent on a nonempty connected domain \(\mathbb{R}\) on the \(z\)-plane such that \(f(z) \neq 0\) for all \(z \in \mathbb{R}\).

**Proof of Theorem 1.6.** Let \(f(z)\) be a Möbius transformation of the form \(u/(z + v), u \neq 0\). Let \(a, b,\) and \(c\) be arbitrary three distinct points in \(\mathbb{R}\) such that
\[
\frac{|a - b|}{|c - b|} = k. \tag{3.1}
\]
We observe that
\[
\frac{a - b}{c - b} \tag{3.2}
\]
is the cross-ratio of \(a, b, c,\) and \(d\), where \(d\) is the point at infinity. Since \(f(z) = u/(z + v), u \neq 0\), we have \(f(d) = 0\). Since Möbius transformations preserve the cross-ratio, we obtain
\[
\frac{f(a) - f(b)}{f(c) - f(b)} \cdot \frac{f(c)}{f(a)} = \frac{a - b}{c - b} \tag{3.3}
\]
which implies that
\[
\frac{|f(a) - f(b)|}{|f(c) - f(b)|} \cdot \frac{|f(c)|}{|f(a)|} = \frac{|a - b|}{|c - b|} = k. \tag{3.4}
\]
Therefore, any Möbius transformation of the form \(u/(z + v), u \neq 0\) satisfies Property 1.5.

Conversely, let \(x\) be an arbitrary fixed point in \(\mathbb{R}\). Then there exists a positive real number \(r\) such that the \(r\) circular neighborhood \(N_r(x)\) of \(x\) is contained in \(\mathbb{R}\).

Throughout the proof let \(A = x + ky, B = x, C = x - y\). Since \(\mathbb{R}\) is a nonempty connected domain on the \(z\)-plane, there exists a positive real number \(s\) such that if
\[
0 < |y| < s, \tag{3.5}
\]
then \(A, B,\) and \(C\) are contained in \(N_r(x)\).
Since \( w = f(z) \) is univalent in \( \mathbb{R} \), \( f(A) = f(x + ky) \), \( f(B) = f(x) \), and \( f(C) = f(x - y) \) are distinct points. By assumption, we have

\[
\left| \frac{f(x + ky) - f(x)}{f(x - y) - f(x)} \cdot \frac{f(x - y)}{f(x + ky)} \right| = k
\]

for all \( y \) such that \( 0 < |y| < s \).

Let

\[
h(y) = \frac{f(x + ky) - f(x)}{f(x - y) - f(x)} \cdot \frac{f(x - y)}{f(x + ky)}.
\]

Then

\[
|h(y)| = k
\]

for all \( y \) such that \( 0 < |y| < s \). The function \( h(y) \) extends analytically at zero by \( h(0) = -k \). Hence, by the maximum modulus principle, we have \( h(y) = -k \) for all \( y \) with \( |y| < s \). In other words, we have

\[
\frac{f(x + ky) - f(x)}{f(x - y) - f(x)} \cdot \frac{f(x - y)}{f(x + ky)} = -k
\]

in \( |y| < s \). This equality implies that

\[
(f(x + ky) - f(x))(f(x - y) - f(x)) = -k(f(x - y) - f(x))f(x + ky).
\]

Differentiate this equality twice with respect to \( y \) and then set \( y = 0 \), we obtain

\[
-k(k + 1)(2(f'(x))^2 - f(x)f''(x)) = 0
\]

which implies that

\[
2(f'(x))^2 - f(x)f''(x) = 0
\]

or

\[
\frac{f(x)f''(x)}{(f'(x))^2} = 2.
\]

By the identity theorem and Lemma 2.5, we conclude that \( f \) is a Möbius transformation of the form \( u/(z + v) \), \( u \neq 0 \).

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