SEQUENTIAL COMPLETENESS OF INDUCTIVE LIMITS

CLAUDIA GÓMEZ and JAN KUČERA

(Received 28 July 1999)

ABSTRACT. A regular inductive limit of sequentially complete spaces is sequentially complete. For the converse of this theorem we have a weaker result: if $\text{ind} E_n$ is sequentially complete inductive limit, and each constituent space E_n is closed in $\text{ind} E_n$, then $\text{ind} E_n$ is α -regular.

Keywords and phrases. Regularity, α -regularity, sequential completeness of locally convex inductive limits.

2000 Mathematics Subject Classification. Primary 46A13; Secondary 46A30.

1. Introduction. In [2] Kučera proves that an LF-space is regular if and only if it is sequentially complete. In [1] Bosch and Kučera prove the equivalence of regularity and sequential completeness for bornivorously webbed spaces. It is natural to ask: can this result be extended to arbitrary sequentially complete spaces? We give a partial answer to this question in this paper.

2. Definitions. Throughout the paper $\{(E_n, \tau_n)\}_{n \in \mathbb{N}}$ is an increasing sequence of Hausdorff locally convex spaces with continuous identity maps $\text{id} : E_n \rightarrow E_{n+1}$ and $E = \text{ind} E_n$ its inductive limit.

A topological vector space E is sequentially complete if every Cauchy sequence $\{x_n\} \subset E$ is convergent to an element $x \in E$.

An inductive limit $\text{ind} E_n$ is regular, resp. α -regular, if each of its bounded sets is bounded, resp. contained, in some constituent space E_n .

3. Main results

THEOREM 3.1. *Let each space (E_n, τ_n) be sequentially complete and $\text{ind} E_n$ regular. Then $\text{ind} E_n$ is sequentially complete.*

PROOF. Let $\{x_k\}$ be a Cauchy sequence in $\text{ind} E_n$. Then the set $\bigcup \{x_k; k \in \mathbb{N}\}$ is bounded in $\text{ind} E_n$ and since the inductive limit is regular, we may assume that it is also bounded in E_1 . For every $n \in \mathbb{N}$, denote by P_n the τ_1 -closure of the convex hull of $\bigcup \{x_k; k > n\}$. Let Q be τ_1 -closed balanced hull of P_0 and F the span of Q with the topology generated by the filter basis $\{\lambda Q; \lambda > 0\}$. Denote this topology by μ .

Before we finish the proof, we prove two lemmas.

LEMMA 3.2. *(F, μ) is Banach.*

PROOF. Let $\{y_k\}$ be a Cauchy sequence in F . Since the set Q is bounded in E_1 , each μ -neighborhood λQ is absorbed by any τ_1 -neighborhood of zero in E_1 . This implies that the topology μ is finer than the topology of F inherited from E_1 . Hence $\{y_k\}$ is Cauchy in E_1 and as such, it converges to some $y \in E_1$ in the topology τ_1 .

For any μ -neighborhood λQ of zero, there exists $k \in \mathbb{N}$ such that $y_p - y_q \in \lambda Q$ for any $p, q > k$. If we let $q \rightarrow \infty$, the τ_1 -closedness of λQ implies that $y_p - y \in \lambda Q$ for any $p > k$, that is $y \in y_p + \lambda Q \subset F$ and $y_p \rightarrow y$ in the topology μ . $\square$

LEMMA 3.3. *The respective families of τ -bounded and μ -bounded sets in the space (F, μ) are the same.*

PROOF. Since the topology μ is finer than τ_1 , which in turn is finer than τ , any μ -bounded set is τ -bounded.

Let $A \subset F$ be a τ -bounded set. Denote by B the μ -closure of the balanced convex hull of A . Put $G = \bigcup \{nB; n \in \mathbb{N}\}$ and equip it with the topology, which we denote by γ , generated by the filter basis $\{\lambda B; \lambda > 0\}$. It follows from Lemma 3.2, that (G, γ) is a Banach space.

Let P_0 and Q be the same sets as above, we first show that the set $Q \cap G$ is closed in (G, γ) . Let a sequence $\{z_k\} \subset Q \cap G$ be convergent to an element $z \in G$ with respect to the topology γ . Since γ is finer than τ_1 , $z_k \rightarrow z$ also in the topology τ_1 and since Q is τ_1 -closed, we have $z \in Q$ and $z \in Q \cap G$. Similarly, the sets $Q_n = n(Q \cap G)$, $n \in \mathbb{N}$, are closed in (G, γ) and $G = \bigcup \{Q_n; n \in \mathbb{N}\}$. By Baire's category theorem, there exists $n \in \mathbb{N}$ such that Q_n has nonempty interior and $Q_n - Q_n = Q_{2n}$ is a γ -neighborhood of zero. Since B is γ -bounded and Q_{2n} is a γ -neighborhood of zero, thus there exists $\beta > 0$ such that $B \subset \beta Q_{2n} \subset \beta 2nQ$, that is, B is μ -bounded. $\square$

To continue the proof of the theorem, we observe that the weak topology σ on F is the weakest topology on F for which the family of all σ -bounded sets in F is the same as the family of all μ -bounded sets in F . So we have $\mu \supset \tau \supset \sigma$ and the τ -Cauchy sequence $\{x_k\}$ is also σ -Cauchy.

Let F'' be the strong second dual of F . Then F can be considered as a closed subspace of F'' . Hence each $f \in F'$ can be continuously extended to F'' and the μ -closed convex set $P_0 \subset F''$ is also $\sigma(F'', F')$ -closed in F'' . Moreover, P_0 as a set bounded in F'' , is equicontinuous on F' . Hence, by Alaoglu's theorem, it is relatively $\sigma(F'', F')$ -compact. This, together with the $\sigma(F'', F')$ -closedness implies that P_0 is $\sigma(F'', F')$ -compact.

Similarly, as for P_0 , all sets P_n , $n \in \mathbb{N}$, are $\sigma(F'', F')$ -closed, and therefore $\sigma(F'', F')$ -compact. Any finite intersection $\bigcap \{P_n; 0 \leq n \leq m\} = P_m$ is nonempty. Hence there exists $x \in \bigcap \{P_n; n \in \mathbb{N}\} \subset F$.

To show that $\bigcap \{P_n; n \in \mathbb{N}\}$ contains only one element, take $y \in F$, $y \neq x$. Then there exists a τ -neighborhood U of zero such that $y \notin x + U$.

Take a τ -closed, balanced, convex τ -neighborhood V of zero such that $V - V \subset U$. There exists $n \in \mathbb{N}$ such that $x_p - x_q \in V$ for any $p, q \geq n$. This implies that $x_p \in x_n + V$ for $p \geq n$, $P_n \subset x_n + V$, and $x_n \in x + V$.

Finally, $P_n \subset x_n + V \subset x + V + V \subset x + U$. But $y \notin x + U$, hence $y \notin P_n$ and $y \notin \bigcap \{P_n; n \in \mathbb{N}\}$.

This implies the existence of an upper-triangular matrix $\Lambda = (\lambda_{nm})$ with all $\lambda_{nm} \geq 0$, only a finite number of nonzero entries in each row, and the sum of all entries in each row is equal to 1, such that the sequence

$$\left\{ w_n = \sum_{m=n}^{\infty} \lambda_{nm} x_m : n \in \mathbb{N} \right\} \quad (3.1)$$

converges to x in the topology of F . Then $w_n \rightarrow x$ also in the weaker topology τ .

Given a balanced convex τ -neighborhood U of zero, there exist $p, q \in \mathbb{N}$ such that $w_n - x \in U$ for $n \geq p$ and $x_m - x_n \in U$ for $m \geq n \geq q$. Then for $n \geq \max(p, q)$, we have

$$x - x_n = (x - w_m) + (w_n - x_n) = (x - w_m) + \sum_{m=n}^{\infty} \lambda_{nm} (x_m - x_n) \in U + U, \quad (3.2)$$

that is, $x_n \rightarrow x$ in E . □

We have proved a little more: *if each E_n is sequentially complete, then any Cauchy sequence in $E = \text{ind } E_n$ which is bounded in some E_n , converges to an element in E_n in the topology inherited from E , but not necessarily in the topology of E_n .*

In [3], Kučera and McKennon constructed a regular quasi-incomplete inductive limit of Banach spaces and they asked about the existence of a sequentially incomplete regular inductive limit of Banach spaces. Theorem 3.1 provides a negative answer.

We do not know whether sequentially complete inductive limit of sequentially complete spaces is regular. Nevertheless, we can at least claim the following.

THEOREM 3.4. *Suppose E_n is closed in $\text{ind } E_n$ for every $n \in \mathbb{N}$ and $\text{ind } E_n$ is sequentially complete. Then $\text{ind } E_n$ is α -regular.*

PROOF. Let $B \subset \text{ind } E_n$ be balanced, convex, closed and bounded. Then the space E_B , with the topology generated by the Minkowski functional of B , is Banach (see Lemma 3.2). Put $B_n = B \cap E_n$, $F_n = E_B \cap E_n$, $n \in \mathbb{N}$, and equip F_n with the topology generated by B_n .

Since E_n is closed in $\text{ind } E_n$ and B_n is closed in E_n , F_n is a Banach subspace of E_B .

We have $E_B = \bigcup \{F_n; n \in \mathbb{N}\}$, hence by the Baire's Category theorem, there exists $n \in \mathbb{N}$ such that F_n contains an open set of E_B . This implies that B_n absorbs B and B is contained in E_n , i.e., $\text{ind } E_n$ is α -regular. □

REFERENCES

- [1] C. Bosch and J. Kucera, *Sequential completeness and regularity of inductive limits of webbed spaces*, to appear in Czechoslovak Math. J.
- [2] J. Kucera, *Sequential Completeness of LF-spaces*, to appear in Czechoslovak Math. J.
- [3] J. Kucera and K. McKennon, *Quasi-incomplete regular LB-space*, Internat. J. Math. Math. Sci. **16** (1993), no. 4, 675–678. MR 94i:46009. Zbl 815.46005.

CLAUDIA GÓMEZ: DEPARTMENT OF PURE AND APPLIED MATHEMATICS, WASHINGTON STATE UNIVERSITY, PULLMAN, WA 99164-3113, USA

E-mail address: gomez@wsu.edu

JAN KUČERA: DEPARTMENT OF PURE AND APPLIED MATHEMATICS, WASHINGTON STATE UNIVERSITY, PULLMAN, WA 99164-3113, USA

E-mail address: kucera@wsu.edu