A NOTE ON M -IDEALS IN CERTAIN ALGEBRAS OF OPERATORS

CHONG-MAN CHO and WOO SUK ROH

(Received 7 October 1998)

ABSTRACT. Let $X = (\sum_{n=1}^{\infty} \ell_1^n)_p$, $p > 1$. In this paper, we investigate M -ideals which are also ideals in $L(X)$, the algebra of all bounded linear operators on X . We show that $K(X)$, the ideal of compact operators on X is the only proper closed ideal in $L(X)$ which is both an ideal and an M -ideal in $L(X)$.

Keywords and phrases. Compact operators, ideal, M -ideal.

2000 Mathematics Subject Classification. Primary 46A32.

1. Introduction. Since Alfsen and Effros [1, 2] introduced the notion of an M -ideal in a Banach space, many authors have studied M -ideals in algebras of operators. An interesting problem has been characterizing and finding those Banach spaces X for which $K(X)$, the space of all compact linear operators on X , is an M -ideal in $L(X)$, the space of all continuous linear operators on X [4, 8, 9, 11, 12].

It is known that if X is a Hilbert space, ℓ_p ($1 < p < \infty$) or c_0 , then $K(X)$ is an M -ideal in $L(X)$ [6, 8, 12] while $K(\ell_1)$ and $K(\ell_\infty)$ are not M -ideals in the corresponding spaces of operators [12]. Smith and Ward [12] proved that M -ideals in a complex Banach algebra with identity are subalgebras and that they are two-sided algebraic ideals if the algebra is commutative. They also proved that M -ideals in a C^* -algebra are exactly the two-sided ideals [12]. Later, Cho and Johnson [5] proved that if X is a uniformly convex Banach space, then every M -ideal in $L(X)$ is a left ideal, and if X^* is also uniformly convex, then every M -ideal in $L(X)$ is a two-sided ideal in $L(X)$.

Flinn [7], and Smith and Ward [13] proved that $K(\ell_p)$ is the only nontrivial M -ideal in $L(\ell_p)$ for $1 < p < \infty$. Kalton and Werner [10] proved that if $1 < p, q < \infty$, $X = (\sum_{n=1}^{\infty} \ell_q^n)_p$ with complex scalars, then $K(X)$ is the only nontrivial M -ideal in $L(X)$. In their proof of this fact, Kalton and Werner [13] used the uniform convexity of X and X^* . In this case, M -ideals in $L(X)$ are two-sided closed ideals in $L(X)$ [5].

In this paper, we investigate M -ideals which are also ideals in $L(X)$ for $X = (\sum_{n=1}^{\infty} \ell_1^n)_p$, $1 < p < \infty$. In our case, neither X nor X^* is uniformly convex. Therefore, no relationships between M -ideals and algebraic ideals in $L(X)$ seem to be known. But still we can use Kalton and Werner's proof in [10] without using uniform convexity of X and X^* to prove that $K(X)$ is the only nontrivial M -ideal in $L(X)$ which is also a closed ideal in $L(X)$ (Theorem 3.3). By duality we have the same conclusion for the space $(\sum_{n=1}^{\infty} \ell_\infty^n)_p$, $1 < p < \infty$.

2. Preliminaries. A closed subspace J of a Banach space X is said to be an L -summand (respectively, M -summand) if there exists a closed subspace J' of X

such that X is an algebraic direct sum $X = J \oplus J'$ and satisfies a norm condition $\|j + j'\| = \|j\| + \|j'\|$ (respectively, $\|j + j'\| = \max\{\|j\|, \|j'\|\}$) for all $j \in J$ and $j' \in J'$. In this case, we write $X = J \oplus_1 J'$ (respectively, $X = J \oplus_\infty J'$) and the projection P on X with $\text{rang } J$ is called an L -projection (respectively, an M -projection). A closed subspace J of a Banach space X is an M -ideal in X if the annihilator $J^\perp$ of J is an L -summand in X^* .

Let A be a complex Banach algebra with identity e . The state space S of A is defined to be $\{\phi \in A^* : \phi(e) = \|\phi\| = 1\}$. An element $h \in A$ is said to be Hermitian if $\|e^{i\lambda h}\| = 1$ for all real number λ . Equivalently, h is Hermitian if and only if $\phi(h)$ is real for every $\phi \in S$ [3, page 46].

In what follows, Z always denote a complex Banach space $(\sum_{n=1}^\infty \ell_1^n)_p$, the ℓ_p -sum of ℓ_1^n 's for $1 < p < \infty$. For each n , let $\{e_{nl}\}_{l=1}^n$ be the standard basis of ℓ_1^n . Then these bases string together to form the standard basis $\{e_n\}_{n=1}^\infty$ of Z and each $T \in L(Z)$ has a matrix representation with respect to $\{e_n\}_{n=1}^\infty$. If $T \in L(Z)$ has the matrix whose (i, j) -entry is t_{ij} , then we can write $T = \sum_{i,j \geq 1} t_{ij} e_j \otimes e_i$, where $e_j \otimes e_i$ is the rank 1 map sending e_j to e_i . Observe that $T(e_j)$ forms the j th column vector of the matrix of T and $\|Te_j\| \leq \|T\|$ for all $j = 1, 2, \dots$. If the matrix of T has at most one nonzero entry in each row and column, then $\|T\|$ is the l_∞ -norm of the sequence of nonzero entries.

A number of facts which hold in $L(\ell_p)$, $1 < p < \infty$, still hold in $L(Z)$. If the matrix of $T \in L(Z)$ is a diagonal matrix (t_{ij}) with real diagonal entries, then for each real λ the matrix of $e^{i\lambda T}$ is also a diagonal matrix with diagonal matrix entries $e^{i\lambda t_{ii}}$. Thus $T \in L(Z)$ is Hermitian if the matrix T is a diagonal matrix with real entries.

Flinn [7] proved that if M is an M -ideal in $L(\ell_p)$, $1 < p < \infty$ and h is a Hermitian element in $L(\ell_p)$ with $h^2 = I$, then $hM \subseteq M$ and $Mh \subseteq M$. From this he proved that if h is any diagonal matrix with real entries, then $hM \subseteq M$ and $Mh \subseteq M$. His proof is valid for Z in place of ℓ_p . Thus we have the following.

LEMMA 2.1. *If M is an M -ideal in $L(Z)$ and $h \in L(Z)$ is a diagonal matrix with real entries, then $hM \subseteq M$ and $Mh \subseteq M$.*

The M -ideal structure of $L(X)$ for $X = (\sum_{n=1}^\infty \ell_q^n)_p$, $1 < p, q < \infty$ was studied by Kalton and Werner [10]. Some of their proofs for X are still good for Z . One of them is the following.

LEMMA 2.2. *There is a constant C such that, whenever (k_n) is a sequence of positive integers with $\limsup k_n = \infty$, then $(\sum_{n=1}^\infty \ell_1^{k_n})_p$ is C -isomorphic to $(\sum_{n=1}^\infty \ell_1^n)_p$.*

PROOF. See proof of Lemma 3.1 of [10]. □

We recall that a Banach space X is C -isomorphic to a Banach space Y if there exists an isomorphism T from X onto Y such that

$$\frac{1}{C} \|x\| \leq \|Tx\| \leq C \|x\| \quad (2.1)$$

for every $x \in X$. We use the following lemma which is due to Kalton and Werner [10].

LEMMA 2.3 [10]. *Let X be a Banach space, $\mathcal{T} \subset L(X)$ be a two-sided ideal, and P a projection onto a complemented subspace E of X which is isomorphic to X .*

(a) *If $P \in \mathcal{T}$, then $\mathcal{T} = L(X)$.*

(b) If E is C -isomorphic with X and $\mathcal{T}$ contains an operator T with $\|T - P\| < (C\|P\|^{-1})$, then $\mathcal{T} = L(X)$.

3. M -ideals in $L((\sum_{n=1}^{\infty} \ell_1^n)_p)$. A matrix carpentry used by Flinn [7] to characterize the M -ideal structure in $L(\ell_p)$ can be used to some extent in our case $Z = (\sum_{n=1}^{\infty} \ell_1^n)_p$. The proof of the following lemma is really a minor modification of Flinn's proof in [7].

LEMMA 3.1. *If M is a nontrivial M -ideal in $L(Z)$, then $K(Z) \subseteq M$.*

SKETCH OF THE PROOF. Let us call two positive integers i and j are in the same block if $n(n+1)/2 < i, j \leq (n+1)(n+2)/2$ for some n . Using Lemma 2.1, we can follow Flinn's proof of the second corollary to Lemma 1 in [7]. The only modification is the following: to prove $2^{1/q} < |t_{pl} + t_{kl}| \leq 2^{1/q}$, we consider two cases. If p and k are in a different block, Flinn's proof just run through. If p and k are in the same block, then $2^{1/q} < |t_{pl} + t_{kl}| \leq \|T(e_l)\| \leq 2^{1/q}$. $\square$

The proof of the following lemma is contained in the proof of Theorem 3.3 in [10].

LEMMA 3.2. *If $\mathcal{T}$ is a closed ideal strictly containing $K(Z)$ then $\mathcal{T}$ contains all the operators which factor through ℓ_p .*

The proof of the following theorem is a modification of that of Kalton and Werner [10]. Here we can go around the use of uniform convexity.

THEOREM 3.3. *If $\mathcal{T}$ is a closed ideal and also an M -ideal in $L(Z)$ strictly containing $K(Z)$, then $\mathcal{T} = L(Z)$.*

PROOF. We recall that the standard basis $\{e_{nl}\}_{l=1}^n$ of ℓ_1^n string together to form the standard basis $\{e_n\}_{n=1}^{\infty}$ of Z . If $\{e'_n\}_{n=1}^{\infty}$ is the standard basis of ℓ_p , then the map $e_n \rightarrow e'_n$ gives a contraction from Z to ℓ_p . Since $E = \overline{\text{span}}\{e_{nl}\}_{n=1}^{\infty}$ is isometric to ℓ_p , there exists a norm one operator A from Z to E carrying e_n to e_{n1} via e'_n . Thus A factors through ℓ_p . By Lemma 3.2, $A \in \mathcal{T}$.

Since $\mathcal{T}$ is also an M -ideal, by Proposition 2.3 in [14], there exists a net $(H_\alpha) \subseteq \mathcal{T}$ such that

$$\limsup \|\pm A + (\text{Id} - H_\alpha)\| = 1. \quad (3.1)$$

To simplify subsequent calculations, let us write the standard basis of Z as $\{e_{nl} : n \in \mathbb{N}, 1 \leq l \leq n\}$ and let $\{e_{nl}^* : n \in \mathbb{N}, 1 \leq l \leq n\}$ be the corresponding biorthogonal functionals. Then $Ae_{nl} = e_{ml}$, where $m = (n-1)n/2 + l$.

Given $0 < \varepsilon < 1$,

$$\max_{\pm} \|\pm A + (\text{Id} - H_\alpha)\| < 1 + \varepsilon \quad (3.2)$$

for infinitely many α 's. For such an α and every e_{nl} ,

$$\max_{\pm} \|\pm Ae_{nl} - (\text{Id} - H_\alpha)e_{nl}\| < 1 + \varepsilon. \quad (3.3)$$

Put $\alpha_{kj} = e_{kj}^*(\text{Id} - H_\alpha)e_{nl}$. Then,

$$\begin{aligned} & \max_{\pm} \|\pm Ae_{nl} + (\text{Id} - H_\alpha)e_{nl}\|^p \\ &= \max_{\pm} \|\pm e_{m1} - (\text{Id} - H_\alpha)e_{nl}\|^p \\ &= \left(\max_{\pm} |\alpha_{m1} \pm 1| + |\alpha_{m2}| + \cdots + |\alpha_{mm}| \right)^p + \sum_{k \neq m} \left(\sum_{j=1}^k |\alpha_{kj}| \right)^p \quad (3.4) \\ &< (1 + \varepsilon)^p. \end{aligned}$$

Since $\max_{\pm} |\alpha_{m1} \pm 1| \geq 1$, it follows that $\sum_{k \neq m} (\sum_{j=1}^k |\alpha_{kj}|)^p < (1 + \varepsilon)^p - 1$ and $|\alpha_{m2}| + \cdots + |\alpha_{mm}| < \varepsilon$. Since $\sqrt{1 + |\alpha_{m1}|^2} \leq \max_{\pm} |\alpha_{m1} \pm 1| < 1 + \varepsilon$, $|\alpha_{m1}| < \sqrt{2\varepsilon + \varepsilon^2} < 2\sqrt{\varepsilon}$. Thus $\|(\text{Id} - H_\alpha)e_{nl}\| < ((3\sqrt{\varepsilon})^p + (1 + \varepsilon)^p - 1)^{1/p} \rightarrow 0$ as $\varepsilon \rightarrow 0$ uniformly in n and l . It follows that, for any n ,

$$\|P_n(\text{Id} - H_\alpha)P_n\| \leq \|P_n(\text{Id} - H_\alpha)j_n\| \leq ((3\sqrt{\varepsilon})^p + (1 + \varepsilon)^p - 1)^{1/p}, \quad (3.5)$$

where P_n is the projection on Z with range $\ell_1^n \subseteq Z$ and j_n is the canonical injection of ℓ_1^n into Z .

By Lemma 3.2 in [10], there exists a sequence (k_n) such that, for the canonical projection P from Z onto $(\sum_{n=1}^\infty \ell_1^{k_n})_p$,

$$\|P - PH_\alpha P\| = \|P(\text{Id} - H_\alpha)P\| < 3((3\sqrt{\varepsilon})^p + (1 + \varepsilon)^p - 1)^{1/p}. \quad (3.6)$$

Since $PH_\alpha P \in \mathcal{T}$ and $\varepsilon > 0$ is arbitrary small, by Lemmas 2.2 and 2.3, $\mathcal{T} = L(Z)$. $\square$

From Lemma 3.1 and Theorem 3.3, we have the following.

COROLLARY 3.4. *If $\mathcal{T}$ is a proper ideal and also an M -ideal in $L(Z)$, then $\mathcal{T} = K(Z)$.*

REMARK. By duality, all the lemmas, Theorem 3.3 and Corollary 3.4 hold with $Z^* = (\sum_{n=1}^\infty \ell_\infty^n)_p$, $1 < p < \infty$, in place of Z .

ACKNOWLEDGEMENT. This research is supported by KOSEF No. 951-0102-003-2.

REFERENCES

- [1] E. M. Alfsen and E. G. Effros, *Structure in real Banach spaces. I*, Ann. of Math. (2) **96** (1972), 98-128. MR 50#5432. Zbl 248.46019.
- [2] ———, *Structure in real Banach spaces. II*, Ann. of Math. (2) **96** (1972), 129-173. MR 50#5432. Zbl 248.46019.
- [3] F. F. Bonsall and J. Duncan, *Numerical Ranges of Operators on Normed Spaces and of Elements of Normed Algebras*, Cambridge University Press, London, 1971. MR 44#5779. Zbl 207.44802.
- [4] C. M. Cho and W. B. Johnson, *A characterization of subspaces X of l_p for which $K(X)$ is an M -ideal in $L(X)$* , Proc. Amer. Math. Soc. **93** (1985), no. 3, 466-470. MR 86h:46026. Zbl 555.47016.
- [5] ———, *M -ideals and ideals in $L(X)$* , J. Operator Theory **16** (1986), no. 2, 245-260. MR 88c:47093. Zbl 623.46022.
- [6] J. Dixmier, *Les fonctionnelles linéaires sur l'ensemble des opérateurs bornés d'un espace de Hilbert*, Ann. of Math. (2) **51** (1950), 387-408. MR 11,441e. Zbl 036.35801.

- [7] P. Flinn, *A characterization of M -ideals in $B(l_p)$ for $1 < p < \infty$* , Pacific J. Math. **98** (1982), no. 1, 73–80. MR 84i:47064. Zbl 469.46036.
- [8] J. Hennefeld, *A decomposition for $B(X)^*$ and unique Hahn-Banach extensions*, Pacific J. Math. **46** (1973), 197–199. MR 51#6492. Zbl 272.46013.
- [9] N. J. Kalton, *M -ideals of compact operators*, Illinois J. Math. **37** (1993), no. 1, 147–169. MR 94b:46028. Zbl 824.46029.
- [10] N. J. Kalton and D. Werner, *The M -ideal structure of some algebras of bounded linear operators*, Proc. Roy. Soc. Edinburgh Sect. A **125** (1995), no. 3, 493–500. MR 97i:46033. Zbl 835.46016.
- [11] A. Lima, *M -ideals of compact operators in classical Banach spaces*, Math. Scand. **44** (1979), no. 1, 207–217. MR 81c:47047. Zbl 407.46019.
- [12] R. R. Smith and J. D. Ward, *M -ideal structure in Banach algebras*, J. Functional Analysis **27** (1978), no. 3, 337–349. MR 57#7175. Zbl 369.46044.
- [13] ———, *Applications of convexity and M -ideal theory to quotient Banach algebras*, Quart. J. Math. Oxford Ser. (2) **30** (1979), no. 119, 365–384. MR 80h:46071. Zbl 412.46042.
- [14] D. Werner, *M -ideals and the “basic inequality”*, J. Approx. Theory **76** (1994), no. 1, 21–30. MR 95i:47080. Zbl 797.41019.

CHO: DEPARTMENT OF MATHEMATICS, HANYANG UNIVERSITY, SEOUL 133-791, KOREA
E-mail address: cmcho@email.hanyang.ac.kr

ROH: DEPARTMENT OF MATHEMATICS, HANYANG UNIVERSITY, SEOUL 133-791, KOREA