A NOTE ON CONNECTEDNESS IN INTUITIONISTIC FUZZY SPECIAL TOPOLOGICAL SPACES

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(Received 31 July 1998)

ABSTRACT. We prove some properties of several types of connectedness defined in intuitionistic fuzzy special topological spaces.

Keywords and phrases. Intuitionistic fuzzy special set, intuitionistic fuzzy special topological space, connectedness.

2000 Mathematics Subject Classification. 54A99, 03E99.

1. Introduction. After the introduction of the concept of fuzzy sets by Zadeh [12], several researches were conducted on the generalizations of the notion of fuzzy set. The idea of "intuitionistic fuzzy set" was first given by Atanassov [2, 3]. Later this concept is generalized to intuitionistic sets in Çoker [6] and intuitionistic topological spaces in [5, 9, 10]. An introduction to connectedness in these spaces is given in [10].

2. Preliminaries. First we present the fundamental definitions (see [6]).

DEFINITION 2.1 (cf. [5, 9]). Let \( X \) be a nonempty fixed set. An intuitionistic fuzzy special set (IFSS for short) \( A \) is an object having the form \( A = \langle x, A_1, A_2 \rangle \), where \( A_1 \) and \( A_2 \) are subsets of \( X \) satisfying \( A_1 \cap A_2 = \emptyset \). The set \( A_1 \) is called the set of members of \( A \), while \( A_2 \) is called the set of nonmembers of \( A \).

The reader may consult [6, 9] to see several types of relations and operations on IFSS's, and intuitionistic fuzzy special points (IFSP's for short) and vanishing intuitionistic fuzzy special points (VIFSP's for short).

DEFINITION 2.2 (cf. [5, 7, 8, 9, 10, 11]). An intuitionistic fuzzy special topology (IFST for short) on a nonempty set \( X \) is a family \( \tau \) of IFSS's in \( X \) containing \( \emptyset, X \) and closed under finite infima and arbitrary suprema. In this case the pair \( (X, \tau) \) is called an intuitionistic fuzzy special topological space (IFSTS for short) and any IFSS in \( \tau \) is known as an intuitionistic fuzzy special open set (IFSOS for short) in \( X \). The complement \( \bar{A} \) of an IFSOS \( A \) in an IFSTS \( (X, \tau) \) is called an intuitionistic fuzzy special closed set (IFSCS for short) in \( X \).

Using a similar construction as in [7], one can easily define the interior and closure operators in IFSTS's.

3. Types of connectedness in intuitionistic fuzzy special topological spaces. Throughout this section \((X, \tau)\) and \((Y, \Phi)\) will always denote IFSTS's. Here we define several types of connectedness in IFSTS's.
Notice that two IFSS’s $A$ and $B$ in $(X, \tau)$ are said to be weakly separated, if $\text{cl}(A) \subseteq \bar{B}$ and $\text{cl}(B) \subseteq \bar{A}$; and $q$-separated, if $\text{cl}(A) \cap B = \emptyset = A \cap \text{cl}(B)$.

**Lemma 3.1.**

\[ A \cap B = \emptyset \implies A \subseteq \bar{B}; \quad (3.1) \]
\[ A \notin \bar{B} \implies A \cap B \neq \emptyset. \quad (3.2) \]

**Definition 3.1** (cf. [1, 10, 11]). Let $(X, \tau)$ be an IFSTS in $X$.

(a) $X$ is called $C_\delta$-disconnected, if there exist weakly separated nonzero IFSS’s $A$ and $B$ in $(X, \tau)$ such that $\bar{X} = A \cup B$. $(X, \tau)$ is called $C_\delta$-connected, if $(X, \tau)$ is not $C_\delta$-disconnected.

(b) $X$ is called $C_M$-disconnected, if there exist $q$-separated nonzero IFSS’s $A$ and $B$ in $(X, \tau)$ such that $\bar{X} = A \cup B$. $X$ is called $C_M$-connected, if $X$ is not $C_M$-disconnected.

The idea of $C_i$-connectedness in fuzzy topological spaces and in intuitionistic fuzzy topological spaces (see [1, 11]) can be generalized to the intuitionistic case.

**Definition 3.2** (cf. [10]). Let $N$ be an IFSS in $(X, \tau)$.

(a) If there exist IFSOS’s $M$ and $W$ in $X$ satisfying the following properties, then $N$ is called $C_i$-disconnected ($i = 1, 2, 3, 4$).

\begin{align*}
C_1: & N \subseteq M \cup W, M \cap W \subseteq \bar{N}, N \cap M \neq \emptyset, N \cap W \neq \emptyset, \\
C_2: & N \subseteq M \cup W, M \cap W = \emptyset, N \cap M \neq \emptyset, N \cap W \neq \emptyset, \\
C_3: & N \subseteq M \cup W, M \cap W \subseteq \bar{N}, M \cap N \neq \emptyset, W \cap \bar{N} = \emptyset, \\
C_4: & N \subseteq M \cup W, M \cap W = \emptyset, M \cap N \neq \emptyset, \bar{W} \subseteq \bar{N}, W \subseteq \bar{N}.
\end{align*}

(b) $N$ is said to be $C_i$-connected ($i = 1, 2, 3, 4$) if $N$ is not $C_i$-disconnected ($i = 1, 2, 3, 4$).

**Corollary 3.1.** $P, Q$ are weakly separated if and only if $\exists M, W \in \tau$ such that $P \subseteq M, Q \subseteq W, P \subseteq \bar{W},$ and $Q \subseteq \bar{M}$.

**Proof.** ($\Leftarrow$) Suppose there exist $M, W \in \tau$ such that $P \subseteq M, Q \subseteq W, P \subseteq \bar{W},$ and $Q \subseteq \bar{M}$. Then $\text{cl}(P) \subseteq \text{cl}(W) = \bar{W}$ (since $\bar{W}$ is an IFSOS) and $\text{cl}(Q) \subseteq \text{cl}(M) = \bar{M} \Rightarrow \text{cl}(P) \subseteq \bar{W} \Rightarrow \text{cl}(P) \subseteq \bar{Q} \Rightarrow \text{cl}(Q) \subseteq \bar{M} \Rightarrow P \Rightarrow \text{cl}(Q) \subseteq \bar{P}$, i.e., $P, Q$ are weakly separated.

($\Rightarrow$) Let $\text{cl}(P) \subseteq \bar{Q}, \text{cl}(Q) \subseteq \bar{P}$. Now take $W = \text{cl}(P)$ and $M = \text{cl}(Q)$ which are IFSOS’s in $(X, \tau)$. Hence $\bar{W} \subseteq \bar{Q}$ and $\bar{M} \subseteq \bar{P} = P \subseteq M, Q \subseteq W$. We also have $W = \text{cl}(P) \subseteq P \Rightarrow P \subseteq \bar{W}$ and $M = \text{cl}(Q) \subseteq Q \Rightarrow Q \subseteq \bar{M}$. 

Here we define $C_\delta$-connectedness and $C_M$-connectedness of an IFSS in $(X, \tau)$.

**Definition 3.3** (cf. Ajmal-Kohli [1]). An IFSS $N$ in $(X, \tau)$ is said to be $C_\delta$-disconnected ($C_M$-disconnected) if and only if there are two nonempty weakly separated ($q$-separated) IFSS’s $A$ and $B$ in $(X, \tau)$ such that $N = A \cup B$. $N$ is called $C_\delta$-connected ($C_M$-connected) if and only if $N$ is not $C_\delta$-disconnected ($C_M$-disconnected).

**Theorem 3.1.** If $N$ is $C_3$-connected, then $N$ is $C_M$-connected.

**Proof.** Let $N$ be $C_M$-disconnected. Then there exist IFSS’s $A, B$ such that $N = A \cup B$, $A, B \neq \emptyset$ and $A, B$ are $q$-separated. Let $P = \text{cl}(A)$ and $Q = \text{cl}(B)$. Then $P, Q$ are IFSOS’s.
Now
\[ \text{cl}(A) \cap \text{cl}(B) \subseteq \bar{A} \cap \bar{B} = \overline{A \cup B} = \bar{N} \Rightarrow N \]
\[ \subseteq \text{cl}(A) \cap \text{cl}(B) = \text{cl}(A) \cup \text{cl}(B) \]
\[ = P \cup Q \Rightarrow N \subseteq P \cup Q, \]
\[ P \cap Q = \text{cl}(A) \cap \text{cl}(B) = \text{cl}(A) \cup \text{cl}(B) = \text{cl}(A \cup B) \subseteq \overline{A \cup B} = \bar{N} \Rightarrow P \cap Q \subseteq \bar{N}. \]

If \( P \subseteq \bar{N} \), then \( N \subseteq \text{cl}(A) \Rightarrow N \cap B = \emptyset \) (since \( \text{cl}(A) \cap B = \emptyset \)) and \( N \cap B = (A \cup B) \cap B = B = \emptyset \). This is a contradiction. Hence \( P \not\subseteq \bar{N} \) follows. \( Q \not\subseteq \bar{N} \) can be proved similarly.

**Theorem 3.2.** If \( N \) is \( C_1 \)-connected, then \( N \) is \( C_3 \)-connected.

**Proof.** Let \( N \) be \( C_3 \)-disconnected. Then there exist IFSS's \( A, B \) such that \( N = A \cup B \), \( A, B \not\subseteq \emptyset \) and \( A, B \) are weakly separated. Let \( P = \text{cl}(A) \) and \( Q = \text{cl}(B) \). Then \( P, Q \) are IFSS's. We have seen that \( N \subseteq P \cup Q \) and \( P \cap Q \subseteq \bar{N} \). If \( P \cap N = \emptyset \), then \( P \not\subseteq \bar{N} \Rightarrow N \subseteq \bar{P} \Rightarrow N \subseteq \text{cl}(A) \subseteq \bar{B} \Rightarrow N \subseteq \bar{B} \). Since \( N = A \cup B \) and \( A \cup B \subseteq \bar{B} \), we obtain a contradiction. Hence \( P \cap N \neq \emptyset \) follows. Similarly, it can be proved that \( Q \cap N \neq \emptyset \).

**Theorem 3.3.** If \( N \) is \( C_3 \)-connected, then \( N \) is \( C_2 \)-connected.

**Proof.** Suppose, on the contrary, that \( N \) is \( C_2 \)-disconnected. Hence there exist IFSS's \( M, W \) such that \( N \subseteq M \cup W \), \( N \cap M \cap W = \emptyset \), \( N \cap M \neq \emptyset \), \( N \cap W = \emptyset \). Now, take \( P = N \cap M \) and \( Q = N \cap W \). Since \( N \subseteq M \cup W \), we get \( N = N \cap (M \cup W) = (N \cap M) \cup (N \cap W) = P \cup Q \). We show that \( P \) and \( Q \) are weakly separated. Let \( P \subseteq M, Q \subseteq W \). Suppose that \( P \not\subseteq \bar{W} \). Then \( P \cap W \neq \emptyset \Rightarrow (N \cap M) \cap W \neq \emptyset \), a contradiction, in other words \( P \subseteq \bar{W} \) follows. Similarly one can also show that \( Q \subseteq M \). Thus \( P, Q \) are weakly separated, which is a contradiction. Therefore \( N \) is \( C_2 \)-connected.

**Theorem 3.4.** If \( N \) is \( C_3 \)-connected, then \( N \) is \( C_3 \)-connected.

**Proof.** Similar to the previous one.

\( C_3 \)-connectedness does not imply \( C_1 \)-connectedness in general:

**Counterexample 3.1.** Let \( X = \{a, b, c\} \) and \( \tau = \{\emptyset, X, A_1, A_2, A_3\} \) where
\[ A_1 = \langle x, \{c\}, \{a, b\} \rangle, \quad A_2 = \langle x, \{a\}, \{b, c\} \rangle, \quad A_3 = \langle x, \{a, c\}, \{b\} \rangle. \]

If \( N = \langle x, \{a\}, \{b\} \rangle \), then \( N \) is \( C_3 \)-connected, since there exist no two nonempty weakly separated IFSS's \( A, B \not\subseteq \emptyset \) such that \( N = A \cup B \). But \( N \) is \( C_1 \)-disconnected.

If \( N \) is \( C_2 \)-connected (\( C_3 \)-connected), then \( N \) may not be \( C_3 \)-connected.

**Counterexample 3.2.** Let \( X = \{a, b, c, d\} \) and \( \tau = \{\emptyset, X, A_1, A_2, A_3, A_4\} \), where
\[ A_1 = \langle x, \{c\}, \{a, b\} \rangle, \quad A_2 = \langle x, \{a, c\}, \{b\} \rangle, \]
\[ A_3 = \langle x, \{a\}, \{b\} \rangle, \quad A_4 = \langle x, \emptyset, \{a, b\} \rangle. \]
\begin{align}
A = \{x, \emptyset, \{a, b, c\}\}, \quad B = \{x, \{a\}, \{b, c\}\}.
\end{align}

(3.7)

\[\text{C}_2\text{-connectedness does not imply C}_M\text{-connectedness in general as shown below.}\]

**Counterexample 3.3.** Let \(X = \{a, b, c\}\) and \(\tau = \{\emptyset, X, A_1, A_2, A_3, A_4\}\), where

\[A_1 = \{x, \{b\}, \{c\}\}, \quad A_2 = \{x, \{c\}, \{a\}\}, \quad A_3 = \{x, \{b, c\}, \emptyset\}, \quad A_4 = \{x, \emptyset, \{a, c\}\}.\]

(3.8)

\[N = \{x, \{c\}, \{a\}\}\] is \(\text{C}_2\)-connected, but not \(\text{C}_M\)-connected, since \(N\) can be expressed as the join of two nonempty \(q\)-separated IFSS’s

\[A = \{x, \{c\}, \{a, b\}\}, \quad B = \{x, \emptyset, \{a, c\}\}.\]

(3.9)

Similarly, \(\text{C}_M\)-connectedness does not imply \(\text{C}_3\cdot (\text{C}_4\cdot)\)-connectedness in general:

**Counterexample 3.4.** Let \(X = \{a, b, c\}\) and \(\tau = \{\emptyset, X, A_1, A_2, A_3\}\), where

\[A_1 = \{x, \{c\}, \{a, b\}\}, \quad A_2 = \{x, \{a\}, \{b, c\}\}, \quad A_3 = \{x, \{a, c\}, \{b\}\}.\]

(3.10)

Let \(N = \{x, \{a\}, \{b\}\}\). \(N\) is \(\text{C}_M\)-connected, since there exist no two nonempty \(q\)-separated IFSS’s \(A, B \neq \emptyset\) such that \(N = A \cup B\). But \(N\) is \(\text{C}_3\)-disconnected (\(\text{C}_4\)-disconnected). If \(N\) is \(\text{C}_4\)-connected, then \(N\) may not be \(\text{C}_M\)-connected.

**Counterexample 3.5.** Let \(X = \{a, b, c, d\}\) and \(\tau = \{\emptyset, X, A_1, A_2, A_3, A_4\}\), where

\[A_1 = \{x, \{c\}, \{a, b\}\}, \quad A_2 = \{x, \{a\}, \{b\}, \{c\}\}, \quad A_3 = \{x, \{a\}, \{b\}\}, \quad A_4 = \{x, \emptyset, \{a, b\}\}.\]

(3.11)

If \(N = \{x, \{a\}, \{b, d\}\}\), then \(N\) is \(\text{C}_4\)-connected, but not \(\text{C}_M\)-connected. This is because, \(N\) can be expressed as the join of two nonempty \(q\)-separated IFSS’s \(A\) and \(B\), where

\[A = \{x, \emptyset, \{a, b, d\}\}, \quad B = \{x, \{a\}, \{b, c, d\}\}.\]

(3.12)

Now, we summarize the relations between several types of connectedness.

\begin{align}
\text{C}_1\text{-connectedness} & \quad \quad \rightarrow \quad \quad \text{C}_3\text{-connectedness} & \quad \quad \rightarrow \quad \quad \text{C}_2\text{-connectedness} \\
\text{C}_M\text{-connectedness} & \quad \quad \quad \quad \leftarrow \quad \quad \text{C}_3\text{-connectedness} & \quad \quad \rightarrow \quad \quad \text{C}_4\text{-connectedness.}
\end{align}

(3.13)

None of these implications are reversible, as given here and in [10]. The following example shows that the closure of \(\text{C}_1\cdot (\text{C}_2\cdot)\)-connected IFSS need not be \(\text{C}_1\)-connected (\(\text{C}_2\)-connected).
**Counterexample 3.6.** Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, A_1, A_2, A_3\}$, where

\[ A_1 = \langle x, \{a, b\}, \{c, d\} \rangle, \quad A_2 = \langle x, \{d\}, \{a, b\} \rangle, \quad A_3 = \langle x, \{a, b, d\}, \emptyset \rangle. \]  

(3.14)

If $N = \langle x, \{a\}, \{c, d\} \rangle$, then $N$ is $C_1$-connected ($C_2$-connected), but $\text{cl}(N)$ is $C_1$-disconnected ($C_2$-disconnected).

**Theorem 3.5.** The closure of $C_3$-connected ($C_4$-connected) IFSS is $C_3$-connected ($C_4$-connected)

**Proof.** Let $N$ be $C_3$-connected, but $\text{cl}(N)$ be $C_3$-disconnected. Hence there exist IFSOS’s $M$, $W \neq \emptyset$ such that $\text{cl}(N) \subseteq M \cup W$, $M \cap W \subseteq \text{cl}(N)$, $M \not\subseteq \text{cl}(N)$, $W \not\subseteq \text{cl}(N)$. We easily deduce $N \subseteq \text{cl}(N) \subseteq M \cup W$ and $M \cap W \subseteq \text{cl}(N) \subseteq \bar{N}$. Since $N$ is $C_3$-connected, $M \subseteq \bar{N}$ or $W \subseteq \bar{N}$ follows. If $M \subseteq \bar{N}$, then $N \subseteq \bar{M} \Rightarrow \text{cl}(N) \subseteq \text{cl}(\bar{M}) = \bar{\text{int}(M)} = \bar{M}$, i.e., $\text{cl}(N) \subseteq \bar{M}$ or $M \subseteq \text{cl}(N)$. But this is a contradiction to the fact $M \not\subseteq \text{cl}(N)$. Similarly, we obtain a contradiction in case $W \subseteq \bar{N}$. Therefore $\text{cl}(N)$ is also $C_3$-connected. The other case can be proved similarly.

**Theorem 3.6.** If $N$ is $C_3$-connected ($C_4$-connected) IFSS in $(X, \tau)$ and $N \subseteq P \subseteq \text{cl}(N)$, then $P$ is $C_3$-connected ($C_4$-connected) IFSS in $(X, \tau)$, too.

**Proof.** Assume the contrary and let $M, W$ be IFSOS’s in $X$ such that $N \subseteq P \subseteq M \cup W$, $M \cap W \subseteq P \subseteq \bar{N}$. Since $N$ is $C_3$-connected, $M \subseteq \bar{N}$ or $W \subseteq \bar{N}$ follows. If $M \subseteq \bar{N}$, then $N \subseteq \bar{M} \Rightarrow \text{cl}(N) \subseteq \text{cl}(\bar{M}) = \text{int}(M) = \bar{M}$, i.e., $\text{cl}(N) \subseteq \bar{M}$ or $M \subseteq \text{cl}(N)$. On the other hand, if $N \subseteq \bar{W}$, then $\text{cl}(N) \subseteq \text{cl}(\bar{W}) = \text{int}(W) = \bar{W} \Rightarrow \text{cl}(N) \subseteq \bar{W}$. $P \subseteq \text{cl}(N) \subseteq \bar{M}$ and $P \subseteq \text{cl}(N) \subseteq \bar{W}$. Therefore $P$ is $C_3$-connected.

This theorem fails in the cases of $C_1$- ($C_2$-)connectedness as shown by the following example.

**Counterexample 3.7.** Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, A_1, A_2, A_3\}$, where

\[ A_1 = \langle x, \{a, b\}, \{c, d\} \rangle, \quad A_2 = \langle x, \{d\}, \{a, b\} \rangle, \quad A_3 = \langle x, \{a, b, d\}, \emptyset \rangle. \]  

(3.15)

If $N = \langle x, \{a\}, \{c, d\} \rangle$, then $N$ is $C_2$-connected. If we take the IFSS $P = \langle x, \{a\}, \{d\} \rangle$, then $P$ satisfies the inclusions $N \subseteq P \subseteq \text{cl}(N)$, and $P$ is not $C_2$-connected. On the other hand, if we consider the $C_1$-connected IFSS $N = \langle x, \{b\}, \{c, d\} \rangle$ in $(X, \tau)$, then $P = \langle x, \{b\}, \{d\} \rangle$ satisfies the inclusions $N \subseteq P \subseteq \text{cl}(N)$, but it is not $C_1$-connected.

**Theorem 3.7.** If $N_1$ and $N_2$ are intersecting $C_1$-connected IFSS’s, then $N_1 \cup N_2$ is also $C_1$-connected.

**Proof.** Assume that $N_1 \cup N_2$ is $C_1$-disconnected. Thus there exist IFSOS’s $M$ and $W$ such that $N_1 \cup N_2 \subseteq M \cup W$ and $M \cap W \subseteq N_1 \cup N_2$, $(N_1 \cup N_2) \cap M \neq \emptyset$ and $(N_1 \cup N_2) \cap W \neq \emptyset$. Since $N_1$ and $N_2$ are $C_1$-connected, then $(N_1 \cap M = \emptyset$ or $N_1 \cap W = \emptyset$) and $(N_2 \cap M = \emptyset$ or $N_2 \cap W = \emptyset$) follow. Since $N_1 \cap N_2 \neq \emptyset$, $\exists \bar{p} \in (N_1 \cap N_2)$, there exist four cases:

**Case 1.** Let $N_1 \cap M = \emptyset$ and $N_2 \cap M = \emptyset$. In this case we get $(N_1 \cap M) \cup (N_2 \cap M) = (N_1 \cup N_2) \cap M = \emptyset$, a contradiction.
CASE 2. Let $N_1 \cap M = \emptyset$ and $N_2 \cap W = \emptyset$. Then $p \notin M, p \notin W$. But this is impossible, since $p \in N_1 \cup N_2 \subseteq M \cup W$.

CASE 3 AND CASE 4. $N_1 \cap W = \emptyset$ and $N_2 \cap M = \emptyset$, or $N_1 \cap W = \emptyset$ and $N_2 \cap W = \emptyset$. These cases may be treated similarly.

Hence it is seen that $N_1 \cup N_2$ is $C_1$-connected.

**Theorem 3.8.** If $N_1$ and $N_2$ are intersecting $C_2$-connected IFSS's, then $N_1 \cup N_2$ is also $C_2$-connected.

**Proof.** Assume that $N_1 \cup N_2$ is $C_2$-disconnected. Then there exist IFSS's $M$ and $W$ such that $N_1 \cup N_2 \subseteq M \cup W$ and $N_1 \cap N_2 \cap M \cap W = \emptyset$. Let $p \in N_1 \cap N_2$, and since $N_1$ and $N_2$ are $C_2$-connected, then $(N_1 \cap N_2) \subseteq N_1 \cup N_2 \subseteq M \cup W = \emptyset$, a contradiction.

**Case 1.** Let $N_1 \cap M = \emptyset$ and $N_2 \cap M = \emptyset$. Then $(N_1 \cup N_2) \subseteq M \cup W$. We obtain $p \notin M, p \notin W$ a contradiction.

**Case 2.** Let $N_1 \cap M = \emptyset$ and $N_2 \cap W = \emptyset$. Then we obtain $p \notin M, p \notin W$ a contradiction to $p \in N_1 \cup N_2 \subseteq M \cup W$.

**Case 3 and Case 4.** They are similar to the ones given above.

Hence $N_1 \cup N_2$ is $C_2$-connected.

**Definition 3.4.** Two IFSS's $A$ and $B$ are said to be overlapping, if $N_1 \subseteq \overline{N}_2$. Conversely, $N_1$ and $N_2$ are said to be nonoverlapping, if $N_1 \subseteq \overline{N}_2$.

Notice that

$$N_1 \not\subseteq \overline{N}_2 \iff N_1^{(1)} \not\subseteq N_2^{(2)} \text{ or } N_1^{(2)} \not\subseteq N_2^{(1)}$$

$$\iff \exists x (x \in N_1^{(1)} \wedge x \notin N_2^{(2)}) \text{ or } \exists y (y \in N_2^{(1)} \wedge y \notin N_1^{(2)})$$

(3.16)

**Theorem 3.9.** If $N_1$ and $N_2$ are overlapping $C_3$-connected IFSS's, then so is $N_1 \cup N_2$.

**Proof.** Let $N_1 \cup N_2$ be $C_3$-disconnected. Then there exist IFSS's $M$ and $W$ such that $N_1 \cup N_2 \subseteq M \cup W$, $M \cap W \subseteq \overline{N}_1 \cup \overline{N}_2$, $M \not\subseteq \overline{N}_1 \cup \overline{N}_2$, $W \not\subseteq \overline{N}_1 \cup \overline{N}_2$. Since $N_1$ and $N_2$ are overlapping, $M \cap W \subseteq \overline{N}_1 \cup \overline{N}_2$, $M \not\subseteq \overline{N}_1 \cup \overline{N}_2$, $W \not\subseteq \overline{N}_1 \cup \overline{N}_2$. Since $N_1$ and $N_2$ are $C_3$-connected, then we obtain: $(M \subseteq \overline{N}_1$ or $W \subseteq \overline{N}_2)$ and $(M \subseteq \overline{N}_1$ or $W \subseteq \overline{N}_2)$.

**Case 1.** Let $M \subseteq \overline{N}_1$ and $M \subseteq \overline{N}_2$. Then $M \subseteq \overline{N}_1 \cap \overline{N}_2 = \overline{N}_1 \cup \overline{N}_2$, a contradiction to $M \not\subseteq \overline{N}_1 \cup \overline{N}_2$.

**Case 2.** Let $M \subseteq \overline{N}_1$ and $W \subseteq \overline{N}_2$. Now suppose that $\exists x (x \in N_1, x \in N_2)$. From $M \subseteq \overline{N}_1$ and $W \subseteq \overline{N}_2$, we obtain $N_1 \cup N_2 \subseteq M \cup W \subseteq \overline{N}_1 \cup \overline{N}_2 = \overline{N}_1 \cup \overline{N}_2 \Rightarrow N_1 \cup N_2 \subseteq \overline{N}_1 \cup \overline{N}_2 \Rightarrow x \in N_1 \cup N_2$ means a contradiction. Similarly, if $\exists y (y \in N_2, y \in N_1)$, we arrive at a contradiction again.

**Case 3 and Case 4.** They are similar to the previous ones.

Hence it follows that $N_1 \cup N_2$ is also $C_3$-connected.

**Theorem 3.10.** If $N_1$ and $N_2$ are overlapping $C_4$-connected IFSS's, then so is $N_1 \cup N_2$. 


Lemma 3.2. If $N_1$ and $N_2$ are $C_3$-connected IFSS's such that $\emptyset \neq (N_1 \cap N_2)$, then $N_1 \cup N_2$ is $C_3$-connected, too.

Proof. For IFSS $A$, the set $\mathcal{A}$ was defined as $\mathcal{A} = \{x, A_1, A_2\}$ if $A = \{x, A_1, A_2\}$. If $\emptyset \neq N_1 \cap N_2$, then we see that $N_1^{(1)} \cap N_2^{(1)} = \emptyset$, i.e., $\exists x \in N_1^{(1)} \cap N_2^{(1)} \Rightarrow x \in N_1^{(1)}$ and $x \in N_2^{(1)} \Rightarrow x \in N_1$ and $x \notin N_2^{(2)} \Rightarrow x \in N_1$ and $x \notin N_2$, i.e., $N_1$ and $N_2$ are overlapping. Hence, the required result follows from a previous theorem.

Lemma 3.3. If $N_1$ and $N_2$ are $C_4$-connected IFSS's such that $\emptyset \neq (N_1 \cap N_2)$, then $N_1 \cup N_2$ is $C_4$-connected, too.

Now, we give generalized versions of these theorems. Here, a family $(N_i)_{i \in J}$ of IFSS’s is said to be nonoverlapping if and only if for each $i \in J$, $N_i$ and $\cap_{j \neq i} N_j$ are nonoverlapping, i.e., $N_i \cap \cap_{j \neq i} N_j \neq \emptyset$.

Theorem 3.11. Let $(N_i)_{i \in J}$ be a family of $C_1$-connected IFSS's such that $\cap N_j \neq \emptyset$. Then $\cup N_i$ is $C_1$-connected, too.

Proof. Let $N = \cup N_i$ be $C_1$-disconnected. Then there exist IFSS's $M$ and $W$ such that $N \subseteq M \cup W$, $M \cap W \subseteq \bar{N}$, $M \cap W \neq \emptyset$, $N \cap W \neq \emptyset$.

Now consider any index $i_0 \in J$. Since $N_{i_0}$ is $C_1$-connected, we have $N_{i_0} \cap M = \emptyset$ or $N_{i_0} \cap W = \emptyset$. Hence there exist three cases:

Case 1. Let $N_i \cap M = \emptyset$ for each $i \in J$. Then, we may write down $N \cap M = (\cup N_i) \cap M = \cup (N_i \cap M) = \cup \emptyset = \emptyset$, which is a contradiction.

Case 2. Let $\tilde{N} \cap W = \emptyset$ for each $i \in J$. Then we obtain a similar contradiction.

Case 3. Let $N_i \cap M = \emptyset$ for each $i \in J$ and $N_i \cap W = \emptyset$ for each $i \in J$, where $J = J_1 \cup J_2$ and $J_1 \neq \emptyset$, $J_2 \neq \emptyset$. Since $\cap N_j \neq \emptyset$, $\exists p \in \cap N_j$. in this case we get $p \notin M$ and $p \notin W$, which is a contradiction with $p \in N \subseteq M \cup W$. Therefore, $N$ is also $C_1$-connected.

Theorem 3.12. Let $(N_i)_{i \in J}$ be a family of $C_2$-connected IFSS's such that $\cap N_j \neq \emptyset$. Then $\cup N_i$ is $C_2$-connected, too.

Proof. Similar to the previous one.

Theorem 3.13. Let $(N_i)_{i \in J}$ be an overlapping family of $C_3$-connected IFSS's. Then $\cup N_i$ is $C_3$-connected, too.

Proof. Let $N = \cup N_i$ be $C_3$-disconnected. Then there exist IFSS's $M$ and $W$ such that $N \subseteq M \cup W$, $M \cap W \subseteq \bar{N}$, $M \neq \bar{N}$, $W \neq \bar{N}$. Now consider any index $i \in J$. Since $N_i$ is $C_3$-connected, we have $M \subseteq N_i$ or $W \subseteq N_i$. Since $(N_i)$ is an overlapping family, suppose further that $\exists i_0 \in J$ such that

$$\exists x \left( x \in N_{i_0}, x \notin \cap_{j \neq i_0} N_j \right) \quad \text{or} \quad \exists y \left( y \in \cap_{j \neq i_0} N_j, y \in N_{i_0} \right).$$

(3.17)
Hence there exist three cases:

**Case 1.** Let \( M \subseteq \overline{N_i} \) for each \( i \in J \). Then we may write down \( M \subseteq \cap N_i = \cup N_i = N \), which is an obvious contradiction.

**Case 2.** Let \( W \subseteq \overline{N_i} \) for each \( i \in J \). Then we obtain a similar contradiction.

**Case 3.** Let \( M \subseteq \overline{N_i} \) for each \( i \in J_1 \) and \( W \subseteq \overline{N_i} \) for each \( i \in J_2 \), where \( J = J_1 \cup J_2 \) and \( J_1 \neq \emptyset, J_2 \neq \emptyset \). Hence

\[
N \subseteq M \cup W \subseteq \left( \cap_{i \in J_1} \overline{N_i} \right) \cup \left( \cap_{i \in J_2} \overline{N_i} \right) = \left( \cup \overline{N_i} \right) \cup \left( \cup \overline{N_i} \right) = \left( \cup_{i \in J_1} \overline{N_i} \right) \cap \left( \cup_{i \in J_2} \overline{N_i} \right)
\]

(3.18)

follows.

Now, let \( \exists x (\in \cap N_i, x \in \cap_{j \neq i} N_j) \). Since \( x \in N_{i_0} \) and hence \( x \in \cap N_i \), we see that \( x \in \overline{N} \Rightarrow x \in \overline{N_{i_0}} \), a contradiction to \( x \in N_{i_0} \). Secondly, let \( \exists y (\in \cap N_i, y \in N_{i_0}) \). From these data we get \( y \in \cap N_i \) and hence \( y \in \overline{N} \). Without loss of generality, we may assume that the index set \( J \setminus \{i_0\} \) has cardinality greater than 1; in other words, \( \exists i_1 \in J \) such that \( i_1 \neq i_0 \). Thus \( y \in N_{i_1} \) and \( y \in \overline{N_{i_1}} \), an obvious contradiction. Therefore, \( \cap N_i \) is also \( C_3 \)-connected.

**Theorem 3.14.** Let \( \langle N_i \rangle \subseteq J \) be an overlapping family of \( C_4 \)-connected IFSOS’s. Then \( \cup N_i \) is \( C_4 \)-connected, too.

**Proof.** Similar to the above proof.

Now, we show that intuitionistic points are always \( C_i \)-connected, unless \( X \) is one-point space \( (i = 1, 2, 3, 4) \).

**Lemma 3.4.** Let \( (X, \tau) \) be an IFSTS and \( p \in X \). Then

(a) \( p \) is \( C_1 \)-connected.

(b) \( \overline{p} \) is \( C_2 \)-connected.

(c) \( p \) is \( C_3 \)-connected.

(d) \( \overline{p} \) is \( C_4 \)-connected.

**Proof.** (a) Assume the contrary, and let \( p \) be \( C_1 \)-disconnected. Hence there exist IFSOS’s \( M \) and \( W \) such that \( p \subseteq M \cup W, M \cap W \subseteq \overline{p} = \langle x, \{p\}_c, \{p\}_d \rangle, p \cap M \neq \emptyset, p \cap W \neq \emptyset \). Since \( p \cap M \neq \emptyset \), and \( p \cap W \neq \emptyset \), we get \( p \in M \) and \( p \in W \); but from \( M \cap W \subseteq \overline{p} \), we see that \( M_1 \cap W_1 \subseteq \{p\}_c \) and \( M_2 \cup W_2 \supseteq \{p\}_d \), which is impossible. Hence \( p \) is \( C_1 \)-connected.

(c) Assume the contrary, and let \( p \) be \( C_3 \)-disconnected. Hence there exist IFSOS’s \( M \) and \( W \) such that \( p \subseteq M \cup W, M \cap W \subseteq \overline{p} = \langle x, \{p\}_c, \{p\}_d \rangle, M \subseteq \overline{p} \) and \( W \subseteq \overline{p} \). Since \( M \subseteq \overline{p} \) and \( W \subseteq \overline{p} \), we get \( p \in M \) and \( p \in W \); and the same reasoning may be applied in this case, too. Hence \( p \) is \( C_3 \)-connected.

(b) and (d) are similar to the first part.
Lemma 3.5. (a) \( p \) is \( C_2 \)-connected.
(b) \( p \) is \( C_3 \)-connected.
(c) \( p \) is \( C_4 \)-connected.

Proof. (a) Suppose the contrary, i.e., let there exist IFSOS’s \( M \) and \( W \) such that
\( p \subseteq M \cup W, M \cap W \cap p = \emptyset, p \cap M \neq \emptyset, \) and \( p \cap W \neq \emptyset . \) Hence, \( \{p\} \cap M^c \cap W^c = \emptyset, \) \( p \in M^c, \) \( p \in W^c \) follow, which is a contradiction.

(b) Suppose not, i.e., let there exist IFSOS’s \( M \) and \( W \) such that
\( p \subseteq M \cup W, M \cap W \subseteq p^c, M \not\subseteq p, \) and \( W \not\subseteq p. \) Hence \( M_1 \cap W_1 \subseteq \{p\}^c, p \in M_1, p \in W_1, \) a contradiction, i.e., \( p^c \) is \( C_3 \)-connected.

(c) Similar to (a) and (b).

Notice that IFSS \( N = (x, N_1, N_2) \) is called proper if and only if \( N_1 \cup N_2 \neq X. \)

Corollary 3.2. In discrete intuitionistic fuzzy special topological space \((X,I(X))\) any nonempty proper IFSS, \( N \) is \( C_1 \)-disconnected.

Proof. Take \( M := N, W := N \in I(X). \) Then \( N \subseteq N \cup N, N \cap N \subseteq N, N \cap N = \emptyset, \) and \( N \cap N \neq \emptyset \) hold, since, for example
\[
N \cap N = (x, N_1 \cap N_2, N_1 \cup N_2) = (x, \emptyset, N_1 \cup N_2) \neq (x, \emptyset, X) = \emptyset. \quad (3.19)
\]

Corollary 3.3. In discrete intuitionistic fuzzy special topological space \((X,I(X))\) any proper IFSS \( N = (x, N_1, N_2) \), where \( N_1 \neq \emptyset, \) is \( C_2 \)-disconnected.

Proof. Take a point \( p \in X \) such that \( p \not\in N_1^c \) and \( p \not\in N_2^c \) and let \( M := p, W := M \) in this IFST. Then we get \( N \subseteq M \cup W, M \cap W \cap N = \emptyset, N \cap M = \emptyset \) and \( N \cap W \neq \emptyset, \) as required.

References


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