CLASSES OF CONVEX FUNCTIONS

**THOMAS ROSY, B. ADOLF STEPHEN, K. G. SUBRAMANIAN,
 and HERB SILVERMAN**

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ABSTRACT. We investigate a family that connects various subclasses of functions convex in the unit disk. We also look at generalized sequences for this family.

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1. Introduction. Denote by S the family of functions

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (1.1)$$

that are analytic and univalent in the unit disk $\Delta = \{z : |z| < 1\}$ and by K the family of convex functions $f \in S$ for which $\operatorname{Re}(1 + zf''/f') > 0$, $z \in \Delta$. There are several well-known subclasses of K . Robertson in [6] introduced the family $K(\alpha)$ of functions f convex of order α , $0 \leq \alpha < 1$, that satisfy in Δ the inequality $\operatorname{Re}(1 + zf''/f') > \alpha$. Ruscheweyh [8] defined the subclass D of K consisting of functions f for which $\operatorname{Re} f'(z) \geq |zf''(z)|$, $z \in \Delta$. His convolution conjecture [8] for this class is stronger than the (former) Bieberbach conjecture (deBranges' theorem).

Goodman [2] introduced the family $UCV \subset K$ of uniformly convex functions f having the property that for every circular arc γ contained in Δ with center also in Δ , the image arc $f(\gamma)$ is a convex arc. He then gave the two-variable characterization

$$\operatorname{Re} \left[1 + \frac{(z - \zeta)f''(z)}{f'(z)} \right] > 0, \quad (z, \zeta) \in \Delta \times \Delta. \quad (1.2)$$

Ma and Minda [4] and Ronning [7] independently found a more applicable one-variable characterization for UCV, namely

$$\operatorname{Re} \left[1 + \frac{zf''(z)}{f'(z)} \right] \geq \left| \frac{zf''(z)}{f'(z)} \right|, \quad z \in \Delta. \quad (1.3)$$

We may summarize relationships between $K(\alpha)$, D , and UCV.

THEOREM 1.1. (i) $D \not\subset K(\alpha)$, $\alpha > 0$; $K(\alpha) \not\subset D$, $\alpha < 1$.

(ii) $D \not\subset UCV$ and $UCV \not\subset D$.

(iii) $UCV \subset K(1/2)$. See [7].

PROOF OF (i). The function $z + z^2/4 \in D - K(\alpha)$, $\alpha > 0$, and

$$\int_0^z (1-t)^{-2(1-\alpha)} dt \in K(\alpha) - D, \quad \alpha < 1. \quad (1.4)$$

□

PROOF OF (ii). For $z + a_2 z^2 + a_3 z^3 + \dots \in D$ and $z + b_2 z^2 + b_3 z^3 + \dots \in \text{UCV}$, the sharp coefficient bounds $|a_2| \leq A_2 = \sqrt{2} - 1$ and $|a_3| \leq A_3 = 2/3(\sqrt{5} - 2)$ were found in [1], while $|b_2| \leq B_2 = 4/\pi^2$ and $|b_3| \leq B_3 = 8/9\pi^2 + 32/3\pi^4$ were found in [4]. Since $A_2 > B_2$ and $B_3 > A_3$, neither inclusion is possible. □

In this paper, we introduce a family of functions that connects these various subclasses of K . We also relate this new class to the family R of functions $f \in S$ for which $\text{Re } f' > 0$, $z \in \Delta$.

2. The main class. We say that f of the form (1.1) is in $\text{UCD}(\alpha)$, $\alpha \geq 0$, if

$$\text{Re } f'(z) \geq \alpha |zf''(z)|, \quad z \in \Delta. \quad (2.1)$$

Note that $\text{UCD}(0) = R$ and $\text{UCD}(1) = D$. Note further that $\text{UCD}(\alpha) \not\subset S$ if $\alpha < 0$, since $z + (1-\alpha)z^2/2 \in \text{UCD}(\alpha) - S$, $\alpha < 0$.

THEOREM 2.1. $\text{UCD}(\alpha) \subset K(1-1/\alpha)$, $\alpha \geq 1$, and the result is sharp.

PROOF. If $f \in \text{UCD}(\alpha)$, then

$$|f'(z)| \geq \alpha |zf''(z)|, \quad \left| \frac{zf''(z)}{f'(z)} \right| \leq \frac{1}{\alpha}. \quad (2.2)$$

Hence,

$$\text{Re} \left[1 + \frac{zf''(z)}{f'(z)} \right] \geq 1 - \left| \frac{zf''(z)}{f'(z)} \right| \geq 1 - \frac{1}{\alpha}. \quad (2.3)$$

For sharpness, set $f(z) = \int_0^z ((1+ct)/(1-ct)) dt$, $c = \sqrt{1+\alpha^2} - \alpha$. Then $f \in \text{UCD}(\alpha)$ because for $|z| = r < 1$, $\text{Re } f'(z) = (1-c^2r^2)/(|1-cz|^2) \geq \alpha(2cr)/(|1-cz|^2) = \alpha|zf''(z)|$. Note that $\text{Re}[1 + zf''/f'] = \text{Re}[1 + 2cz/(1-c^2z^2)]$. For $z = -r$, $r \rightarrow 1$, this last expression approaches $1 - 2c/(1-c^2) = 1 - 1/\alpha$. Thus, $f \notin K(\beta)$ for $\beta > 1 - 1/\alpha$. □

Clearly, the family $\text{UCD}(\alpha) \subset D$ for $\alpha \geq 1$. We next see when $\text{UCD}(\alpha)$ is uniformly convex.

THEOREM 2.2. $\text{UCD}(\alpha) \subset \text{UCV} \Leftrightarrow \alpha \geq 2$.

PROOF. Since the extremal function of Theorem 2.1 is not in $K(1/2)$ for $\alpha < 2$, an application of Theorem 1.1(iii) shows that this function cannot be in UCV when $\alpha < 2$.

If $f \in \text{UCD}(2)$, then

$$|f'(z)| \geq 2 |zf''(z)|, \quad \left| \frac{zf''(z)}{f'(z)} \right| \leq \frac{1}{2}. \quad (2.4)$$

Thus,

$$\text{Re} \left[1 + \frac{zf''(z)}{f'(z)} \right] \geq 1 - \left| \frac{zf''(z)}{f'(z)} \right| \geq \left| \frac{zf''(z)}{f'(z)} \right|, \quad f \in \text{UCV}. \quad (2.5)$$

□

3. Sequences. To a finite or infinite increasing sequence of integers $\{n_k\}$ with $n_k \geq k$ we associate with f of the form (1.1) the generalized partial sum defined by

$$\tilde{f}(z) = z + \sum_{k=2}^{\infty} a_{n_k} z^{n_k}, \quad (3.1)$$

with the special case $n_k = k$ ($k = 2, 3, \dots, n$) representing the n th section $f_n(z) = z + \sum_{k=2}^n a_k z^k$. We determine when generalized sequences of functions in R satisfy conditions to be in $\text{UCD}(\alpha)$. Since our results rely on properties for continuous linear functionals defined on R , sharp results are obtained from the extreme points of R . See [3]. It thus suffices to consider the extremal function $f \in R$ defined by

$$f(z) = -z - 2 \log(1-z) = z + 2 \sum_{k=2}^{\infty} \frac{z^k}{k}. \quad (3.2)$$

In [10] it was shown for $f \in R$ that

- (i) $4f_n(z/4) \subset D$,
- (ii) $f(az)/a \subset D$, $a = \sqrt{2} - 1$.

The proof of (i) for $|z| = r \leq 1/2$ relied on the inequalities

$$\operatorname{Re} f'_n(z) \geq \frac{(1+r)^2(1-2r)}{|1-z|^2}, \quad |f''_n(z)| \leq \frac{2(1+r)^2}{|1-z|^2}, \quad (3.3)$$

and of (ii) for $r < 1$ on

$$\operatorname{Re} f'(z) \geq \frac{(1-r^2)}{|1-z|^2}, \quad |f''(z)| \leq \frac{2}{|1-z|^2}. \quad (3.4)$$

We extend these results to the class $\text{UCD}(\alpha)$.

THEOREM 3.1. *If $f \in R$, then*

- (i) $f_n(bz)/b \in \text{UCD}(\alpha)$, $b = 1/2(1+\alpha)$,
- (ii) $f(az)/a \in \text{UCD}(\alpha)$, $a = \sqrt{\alpha^2+1} - \alpha$.

The results are sharp for all $\alpha \geq 0$.

PROOF OF (i). From (3.3) we have

$$\operatorname{Re} f'_n(z) \geq \alpha |zf''_n(z)| \quad \text{when } (1+r)^2(1-2r) \geq 2\alpha r(1+r)^2, \quad (3.5)$$

which is true for $r \leq 1/2(1+\alpha)$. Equality holds for f defined by (3.2) and $n = 2$. $\square$

PROOF OF (ii). From (3.4) we see that

$$\operatorname{Re} f'(z) \geq \alpha |zf''(z)| \quad \text{when } 1-r^2 \geq 2\alpha r \quad (3.6)$$

which holds for $r = \sqrt{\alpha^2+1} - \alpha$. $\square$

REMARK 3.2. The case $\alpha = 0$ in (i) ($f \in R$) is due to MacGregor [5].

We now turn to generalized sequences.

THEOREM 3.3. If f of the form (1.1) is in R with $\tilde{f}$ of the form (3.1) a generalized sum of f , then $\tilde{f}(bz)/b \in \text{UCD}(\alpha)$, $\alpha \geq 0$, where b is the positive zero in $(0, 1)$ of

$$1 - 2r - r^2 - 2\alpha r \frac{1+r^2}{1-r^2} = 0. \quad (3.7)$$

The result is sharp for all α .

REMARK 3.4. The cases $\alpha = 0$ ($b = \sqrt{2} - 1$) and $\alpha = 1$ ($b \approx 0.2253$) were proved in [10]. Note that the value for b decreases as α increases.

PROOF. We need only consider f defined by (3.2). Defining h by

$$h'(z) = f'(z) + \alpha e^{iy} z f''(z), \quad y \text{ real}, \quad (3.8)$$

it suffices to show that

$$\tilde{h}'(z) = \tilde{f}'(z) + \alpha e^{iy} z \tilde{f}''(z) = 1 + 2 \sum_{n_k=2}^{\infty} [1 + \alpha(n_k - 1)e^{iy}] z^{n_k-1} \quad (3.9)$$

has positive real part for $|z| < b$. We examine different cases.

CASE 1 ($n_2 \geq 3$). Then

$$\begin{aligned} \text{Re } \tilde{h}'(z) &\geq 1 - 2 \sum_{n=3}^{\infty} [1 + \alpha(n-1)] r^{n-1} = 1 - \frac{2r^2}{1-r} - \frac{2\alpha r^2(2-r)}{(1-r)^2}, \\ (1-r)^2 \text{Re } \tilde{h}'(z) &\geq 1 - 2r - r^2 - 2\alpha r^2(2-r) \geq 1 - 2r - r^2 - 2\alpha r. \end{aligned} \quad (3.10)$$

Since this last expression is bounded below by the left-hand side of (3.7), it follows that

$$\text{Re } \tilde{h}'(z) \geq 0 \quad \text{for } |z| \leq b. \quad (3.11)$$

CASE 2 ($n_2 = 2, n_3 = 3$). Then for $z = r e^{i\theta}$,

$$\begin{aligned} \text{Re } \tilde{h}'(z) &\geq \text{Re} [1 + 2(1 + \alpha e^{iy})z + 2(1 + 2\alpha e^{iy})z^2] - 2 \sum_{n=4}^{\infty} [1 + \alpha(n-1)] r^{n-1} \\ &:= \text{Re } A(z) - \frac{2r^3}{(1-r)^2} [(1-r) + \alpha(3-4r)]. \end{aligned} \quad (3.12)$$

Now

$$\begin{aligned} \text{Re } A(z) &= 1 + 2r \cos \theta + 2r^2 \cos 2\theta + \text{Re} [2\alpha e^{iy}(z + 2z^2)] \\ &\geq 1 + 2r \cos \theta + 2r^2 \cos 2\theta - 2\alpha r(1 + 2r \cos \theta), \end{aligned} \quad (3.13)$$

which attains its minimum for $r = b$ when $\cos \theta = -(1 - 2\alpha b)/4b$. Thus,

$$\begin{aligned} \text{Re } A(z) &\geq \frac{3}{4} - 2b^2 - \alpha b - \alpha^2 b^2 \quad \text{for } |z| \leq b, \\ \text{Re } \tilde{h}'(z) &\geq \frac{3}{4} - 2b^2 - \alpha b - \alpha^2 b^2 - \frac{2b^3}{(1-b)^2} (1 - b + \alpha(3 - 4b)). \end{aligned} \quad (3.14)$$

Substituting from (3.7) the value $\alpha = (1 - b^2)(1 - b^2 - 2b)/2b(1 + b^2)$ into the right-hand side of (3.14), one can show that the right-hand side of (3.14) decreases as b decreases. Since $\alpha b \rightarrow 1/2$ as $\alpha \rightarrow \infty$, we see that $\operatorname{Re} \tilde{h}'(z) \geq 3/4 - 1/2 - 1/4 = 0$, $|z| \leq b$.

When $n_2 = 2$ and $n_3 \geq 4$, we consider two remaining possibilities, depending on whether the first n_k after consecutive even integers is the succeeding odd integer.

CASE 3. We have

$$\begin{aligned} \tilde{h}'(z) = 1 + 2 \sum_{n=1}^{m+1} [1 + \alpha(2n+1)e^{iy}]z^{2n-1} + 2[1 + \alpha(2m+2)e^{iy}]z^{2m+2} \\ + 2 \sum_{n_k \geq 2m+4} [1 + \alpha(n_k-1)e^{iy}]z^{n_k-1}. \end{aligned} \quad (3.15)$$

Setting $r'(z) = h'(z) - \tilde{h}'(z)$, we have for $|z| \leq b$ that

$$\begin{aligned} \operatorname{Re} \tilde{h}'(z) &\geq \frac{1 - b^2 - 2\alpha b}{(1 + b)^2} - |r'(z)| \\ &\geq \frac{1 - b^2 - 2\alpha b}{(1 + b)^2} - 2 \sum_{n=1}^m (1 + 2\alpha n)b^{2n} - 2 \sum_{n=2m+3}^{\infty} (1 + \alpha n)b^n. \end{aligned} \quad (3.16)$$

An induction shows that the right-hand side decreases with m , so that

$$\begin{aligned} \operatorname{Re} \tilde{h}'(z) &\geq \frac{1 - b^2 - 2\alpha b}{(1 + b)^2} - 2 \sum_{n=1}^{\infty} (1 + 2\alpha n)b^{2n} \\ &= \frac{1 - b^2 - 2\alpha b}{(1 + b)^2} - \frac{2b^2(1 + 2\alpha - b^2)}{(1 - b^2)^2} \\ &= \frac{1}{1 - b^2} \left[1 - 2b - b^2 - 2\alpha b \left(\frac{1 + b^2}{1 - b^2} \right) \right] = 0. \end{aligned} \quad (3.17)$$

CASE 4. We have

$$\tilde{h}'(z) = 1 + 2 \sum_{n=1}^m [1 + \alpha(2n-1)e^{iy}]z^{2n-1} + 2 \sum_{n_k \geq 2m+3} [1 + \alpha(n_k-1)]z^{n_k-1}. \quad (3.18)$$

Then for $|z| \leq b$,

$$\operatorname{Re} \tilde{h}'(z) \geq 1 - 2 \sum_{n=1}^m [1 + \alpha(2n-1)]b^{2n-1} - 2 \sum_{n=2m+3}^{\infty} [1 + \alpha(n-1)]b^{n-1}. \quad (3.19)$$

Again the right-hand side decreases with m and

$$\operatorname{Re} \tilde{h}'(z) \geq 1 - 2 \sum_{n=1}^{\infty} [1 + \alpha(2n-1)]b^{2n-1} = 1 - \frac{2b}{1 - b^2} - 2\alpha b \left(\frac{1 + b^2}{(1 - b^2)^2} \right) = 0. \quad (3.20)$$

For sharpness, set $n_k = 2k$ so that $\tilde{f}(z) = z + 2 \sum_{k=1}^{\infty} z^{2k}/2k$. Setting $y = 0$ in (3.13), we see that $\tilde{h}'(-b) = 0$. $\square$

4. Sufficient conditions. We next see how small the coefficient need to be in order to guarantee inclusion in the family.

THEOREM 4.1. *A sufficient condition for f of the form (1.1) to be in $\text{UCD}(\alpha)$, $\alpha \geq 0$, is that $\sum_{k=2}^{\infty} k[1 + \alpha(k-1)]|a_k| \leq 1$.*

PROOF. Since $\text{Re } f' \geq 1 - \sum_{k=2}^{\infty} k|a_k|$ and $|zf''| \leq \sum_{k=2}^{\infty} k(k-1)|a_k|$, the result follows. $\square$

In [9] the family T consisting of univalent functions f of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad a_k \geq 0, \quad (4.1)$$

was investigated. Denote by $\text{TUCD}(\alpha)$ functions in $\text{UCD}(\alpha)$ of the form (4.1). For this class, the sufficient condition of Theorem 4.1 is also necessary.

THEOREM 4.2. *A function of the form (4.1) is in $\text{TUCD}(\alpha)$ if and only if $\sum_{k=2}^{\infty} k[1 + \alpha(k-1)]a_k \leq 1$.*

PROOF. In view of Theorem 4.1, we need only show that $f \in \text{TUCD}(\alpha)$ satisfies the coefficient condition. Note that

$$f'(r) = 1 - \sum_{k=2}^{\infty} k a_k r^{k-1}, \quad \alpha r f''(r) = \alpha \sum_{k=2}^{\infty} k(k-1) r^{k-1}. \quad (4.2)$$

The result follows upon letting $r \rightarrow 1$. $\square$

REMARK 4.3. The coefficient characterizations found in [9] also show that f of the form (4.1) is starlike $\Leftrightarrow f \in \text{TUCD}(0)$, is convex $\Leftrightarrow f \in \text{TUCD}(1)$, and is convex of order $1/2 \Leftrightarrow f \in \text{TUCD}(2)$. A function f of the form (4.1) is also uniformly convex $\Leftrightarrow f \in \text{TUCD}(2)$. See [11].

From the work in [9], the coefficient characterization of Theorem 4.2 enables us to determine extreme points.

THEOREM 4.4. *The extreme points of $\text{TUCD}(\alpha)$ are $f_1(z) = z$ and*

$$f_k(z) = z - \frac{z^k}{k[1 + \alpha(k-1)]}, \quad k = 2, 3, \dots, \quad (4.3)$$

and $f \in \text{TUCD}(\alpha) \Leftrightarrow f$ can be expressed in the form

$$f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z), \quad \text{where } \lambda_k \geq 0, \quad \sum_{k=1}^{\infty} \lambda_k = 1. \quad (4.4)$$

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ROSY, STEPHEN, AND SUBRAMANIAN: DEPARTMENT OF MATHEMATICS, MADRAS CHRISTIAN COLLEGE, TAMBARAM, MADRAS-600 059, INDIA

SILVERMAN: DEPARTMENT OF MATHEMATICS, COLLEGE OF CHARLESTON, CHARLESTON, SC 29424, USA

E-mail address: silvermanh@cofc.edu