COMPLETE CONVERGENCE FOR SUMS OF ARRAYS OF RANDOM ELEMENTS

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ABSTRACT. Let $\{X_{ni}\}$ be an array of rowwise independent B -valued random elements and $\{a_n\}$ constants such that $0 < a_n \uparrow \infty$. Under some moment conditions for the array, it is shown that $\sum_{i=1}^n X_{ni}/a_n$ converges to 0 completely if and only if $\sum_{i=1}^n X_{ni}/a_n$ converges to 0 in probability.

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1. Introduction. Let $(B, \|\cdot\|)$ be a real separable Banach space. A separable Banach space B is said to be of type r , $1 \leq r \leq 2$, if there exists a constants C_r such that

$$E \left\| \sum_{i=1}^n X_i \right\|^r \leq C_r \sum_{i=1}^n E \|X_i\|^r \quad (1.1)$$

for all independent B -valued random elements $X_1, \dots, X_n$ with mean zero and finite r th moments.

A sequence $\{X_n, n \geq 1\}$ of B -valued random elements is said to converge completely to zero if for each $\epsilon > 0$,

$$\sum_{n=1}^{\infty} P(\|X_n\| > \epsilon) < \infty. \quad (1.2)$$

Note that complete convergence implies almost surely by the Borel-Cantelli lemma.

Now let $\{X_n, n \geq 1\}$ be a sequence of independent random variables. Let $\psi(t)$ be a positive, even and continuous function such that

$$\frac{\psi(|t|)}{|t|} \uparrow \text{ and } \frac{\psi(|t|)}{|t|^2} \downarrow \text{ as } |t| \uparrow. \quad (1.3)$$

Chung [3] strong law of large numbers (SLLN) states that if

$$EX_n = 0 \quad \text{for } n \geq 1, \quad \sum_{n=1}^{\infty} \frac{E\psi(|X_n|)}{\psi(n)} < \infty \quad (1.4)$$

then

$$\frac{\sum_{i=1}^n X_i}{n} \rightarrow 0 \text{ almost surely.} \quad (1.5)$$

Recently, Hu and Taylor [6] proved Chung type SLLN for arrays of rowwise independent random variables. More specifically, let $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of rowwise independent random variables and let $\{a_n, n \geq 1\}$ be a sequence of real numbers with $0 < a_n \uparrow \infty$. Let $\psi(t)$ be a positive, even and continuous function such that

$$\frac{\psi(|t|)}{|t|^p} \uparrow \text{ and } \frac{\psi(|t|)}{|t|^{p+1}} \downarrow \text{ as } |t| \uparrow \quad (1.6)$$

for some integer $p \geq 2$. Furthermore, assume that

$$EX_{ni} = 0 \quad \text{for } 1 \leq i \leq n, n \geq 1, \quad (1.7)$$

$$\sum_{n=1}^{\infty} \sum_{i=1}^n \frac{E\psi(|X_{ni}|)}{\psi(a_n)} < \infty, \quad (1.8)$$

$$\sum_{n=1}^{\infty} \left(\sum_{i=1}^n \frac{EX_{ni}^2}{a_n^2} \right)^{2k} < \infty, \quad (1.9)$$

where k is a positive integer. Then the conditions (1.6), (1.7), (1.8), and (1.9) imply

$$\frac{1}{a_n} \sum_{i=1}^n X_{ni} \rightarrow 0 \text{ almost surely.} \quad (1.10)$$

Many classical theorems hold for B -valued random elements under the assumption that the weak law of large numbers (WLLN) holds (see, Kuelbs and Zinn [8], de Acosta [4], Choi and Sung [1, 2], Wang, Rao and Yang [10], Kuczmaszewska and Szynal [7], and Sung [9]).

In this paper, we apply de Acosta [4] inequality to obtain Hu and Taylor's [6] result in a general Banach space under the assumption that WLLN holds.

2. Main Result. To prove our main theorem, we need the following lemma which is due to de Acosta [4].

LEMMA 2.1. *For each $p \geq 1$, there exists a positive constant C_p such that for separable Banach space B and any finite sequence $\{X_i, 1 \leq i \leq n\}$ of independent B -valued random elements with $E\|X_i\|^p < \infty$ ($1 \leq i \leq n$), the following inequalities hold.*

(i) For $1 \leq p \leq 2$,

$$E \left\| \sum_{i=1}^n X_i - E \sum_{i=1}^n X_i \right\|^p \leq C_p \sum_{i=1}^n E\|X_i\|^p. \quad (2.1)$$

(ii) For $p > 2$,

$$E \left\| \sum_{i=1}^n X_i - E \sum_{i=1}^n X_i \right\|^p \leq C_p \left[\left(\sum_{i=1}^n E\|X_i\|^2 \right)^{p/2} + \sum_{i=1}^n E\|X_i\|^p \right]. \quad (2.2)$$

Throughout this paper, let $\psi(t)$ be a positive, even function such that

$$\frac{\psi(|t|)}{|t|} \uparrow \text{ and } \frac{\psi(|t|)}{|t|^p} \downarrow \text{ as } |t| \uparrow \quad (2.3)$$

for some $p \geq 1$.

THEOREM 2.2. Let $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of rowwise independent B -valued random elements and $\{a_n, n \geq 1\}$ constants such that $0 < a_n \uparrow \infty$. Assume that

$$\sum_{n=1}^{\infty} \sum_{i=1}^n \frac{E\psi(\|X_{ni}\|)}{\psi(a_n)} < \infty, \quad (2.4)$$

$$\sum_{n=1}^{\infty} \left(\sum_{i=1}^n \frac{E\|X_{ni}\|^2}{a_n^2} \right)^s < \infty \quad (2.5)$$

for some $s > 0$. Then the following statements are equivalent.

- (i) $(1/a_n) \sum_{i=1}^n X_{ni} \rightarrow 0$ in L^1 .
- (ii) $(1/a_n) \sum_{i=1}^n X_{ni} \rightarrow 0$ completely.
- (iii) $(1/a_n) \sum_{i=1}^n X_{ni} \rightarrow 0$ almost surely.
- (iv) $(1/a_n) \sum_{i=1}^n X_{ni} \rightarrow 0$ in probability.

PROOF. (i) $\Rightarrow$ (ii). Define $Y_{ni} = X_{ni}I(\|X_{ni}\| \leq a_n)$ and $Z_{ni} = X_{ni}I(\|X_{ni}\| > a_n)$. Since $\psi(|t|)/|t|$ is an increasing function of $|t|$, we have by (2.4) that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{a_n} E \left\| \sum_{i=1}^n Z_{ni} \right\| &\leq \sum_{n=1}^{\infty} \frac{1}{a_n} \sum_{i=1}^n E \|Z_{ni}\| \\ &\leq \sum_{n=1}^{\infty} \frac{1}{\psi(a_n)} \sum_{i=1}^n E \psi(\|Z_{ni}\|) \leq \sum_{n=1}^{\infty} \sum_{i=1}^n \frac{E\psi(\|X_{ni}\|)}{\psi(a_n)} < \infty. \end{aligned} \quad (2.6)$$

It follows that

$$\frac{1}{a_n} \sum_{i=1}^n Z_{ni} \rightarrow 0 \text{ completely.} \quad (2.7)$$

The proof will be completed by showing that

$$\frac{1}{a_n} \sum_{i=1}^n Y_{ni} \rightarrow 0 \text{ completely.} \quad (2.8)$$

From (i) and (2.6), we have

$$\begin{aligned} \frac{1}{a_n} E \left\| \sum_{i=1}^n Y_{ni} \right\| &= \frac{1}{a_n} E \left\| \sum_{i=1}^n (X_{ni} - Z_{ni}) \right\| \\ &\leq \frac{1}{a_n} E \left\| \sum_{i=1}^n X_{ni} \right\| + \frac{1}{a_n} E \left\| \sum_{i=1}^n Z_{ni} \right\| \rightarrow 0. \end{aligned} \quad (2.9)$$

Thus, to prove (2.8), it is enough to show that

$$\frac{1}{a_n} \left\| \sum_{i=1}^n Y_{ni} \right\| - \frac{1}{a_n} E \left\| \sum_{i=1}^n Y_{ni} \right\| \rightarrow 0 \text{ completely.} \quad (2.10)$$

First consider the case of $1 \leq p \leq 2$. From Markov's inequality and Lemma 2.1(i), we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} P\left(\left|\frac{1}{a_n}\left\|\sum_{i=1}^n Y_{ni}\right\| - \frac{1}{a_n}E\left\|\sum_{i=1}^n Y_{ni}\right\|\right| > \epsilon\right) \\
& \leq \frac{1}{\epsilon^p} \sum_{n=1}^{\infty} \frac{1}{a_n^p} E\left\|\sum_{i=1}^n Y_{ni}\right\|^p - E\left\|\sum_{i=1}^n Y_{ni}\right\|^p \\
& \leq \frac{C_p}{\epsilon^p} \sum_{n=1}^{\infty} \sum_{i=1}^n \frac{E\|Y_{ni}\|^p}{a_n^p} \leq \frac{C_p}{\epsilon^p} \sum_{n=1}^{\infty} \sum_{i=1}^n \frac{E\psi(\|Y_{ni}\|)}{\psi(a_n)} \\
& \leq \frac{C_p}{\epsilon^p} \sum_{n=1}^{\infty} \sum_{i=1}^n \frac{E\psi(\|X_{ni}\|)}{\psi(a_n)} < \infty,
\end{aligned} \tag{2.11}$$

since $\psi(\|t\|)/|t|^p \downarrow$ and (2.4). Thus (2.10) holds.

Now consider the case of $p > 2$. Note that $\psi(\|t\|)/|t|^p \downarrow$ implies $\psi(\|t\|)/|t|^q \downarrow$ for each $q \geq p$. Let $q = \max\{p, 2s\}$. Then we have by Markov's inequality and Lemma 2.1(ii) that

$$\begin{aligned}
& \sum_{n=1}^{\infty} P\left(\left|\frac{1}{a_n}\left\|\sum_{i=1}^n Y_{ni}\right\| - \frac{1}{a_n}E\left\|\sum_{i=1}^n Y_{ni}\right\|\right| > \epsilon\right) \\
& \leq \frac{1}{\epsilon^q} \sum_{n=1}^{\infty} \frac{1}{a_n^q} E\left\|\sum_{i=1}^n Y_{ni}\right\|^q - E\left\|\sum_{i=1}^n Y_{ni}\right\|^q \\
& \leq \frac{C_q}{\epsilon^q} \sum_{n=1}^{\infty} \frac{1}{a_n^q} \left[\left(\sum_{i=1}^n E\|Y_{ni}\|^2 \right)^{q/2} + \sum_{i=1}^n E\|Y_{ni}\|^q \right] \\
& = \frac{C_q}{\epsilon^q} \sum_{n=1}^{\infty} \left(\frac{\sum_{i=1}^n E\|Y_{ni}\|^2}{a_n^2} \right)^{q/2} + \frac{C_q}{\epsilon^q} \sum_{n=1}^{\infty} \frac{1}{a_n^q} \sum_{i=1}^n E\|Y_{ni}\|^q.
\end{aligned} \tag{2.12}$$

Since $q \geq p$, $\psi(\|t\|)/|t|^p \downarrow$ implies $\psi(\|t\|)/|t|^q \downarrow$, and so

$$\sum_{n=1}^{\infty} \frac{1}{a_n^q} \sum_{i=1}^n E\|Y_{ni}\|^q \leq \sum_{n=1}^{\infty} \sum_{i=1}^n \frac{E\psi(\|Y_{ni}\|)}{\psi(a_n)} \leq \sum_{n=1}^{\infty} \sum_{i=1}^n \frac{E\psi(\|X_{ni}\|)}{\psi(a_n)} < \infty. \tag{2.13}$$

Also,

$$\begin{aligned}
& \sum_{n=1}^{\infty} \left(\frac{\sum_{i=1}^n E\|Y_{ni}\|^2}{a_n^2} \right)^{q/2} \leq \left[\sum_{n=1}^{\infty} \left(\frac{\sum_{i=1}^n E\|Y_{ni}\|^2}{a_n^2} \right)^s \right]^{q/2s} \\
& \leq \left[\sum_{n=1}^{\infty} \left(\frac{\sum_{i=1}^n E\|X_{ni}\|^2}{a_n^2} \right)^s \right]^{q/2s} < \infty,
\end{aligned} \tag{2.14}$$

since $q \geq 2s$ and (2.5). Combining (2.12), (2.13), and (2.14) yields (2.10). Thus (i) $\Rightarrow$ (ii) is proved. Since the implications (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv) are obvious, it remains to show that (iv) $\Rightarrow$ (i).

Assume that (iv) holds. From Lemma 2.1(i) and (2.5)

$$E\left\|\frac{1}{a_n}\left\|\sum_{i=1}^n X_{ni}\right\| - \frac{1}{a_n}E\left\|\sum_{i=1}^n X_{ni}\right\|\right\|^2 \leq \frac{C_2}{a_n^2} \sum_{i=1}^n E\|X_{ni}\|^2 \rightarrow 0, \tag{2.15}$$

which entails

$$\frac{1}{a_n}\left\|\sum_{i=1}^n X_{ni}\right\| - \frac{1}{a_n}E\left\|\sum_{i=1}^n X_{ni}\right\| \rightarrow 0 \text{ in probability.} \tag{2.16}$$

It follows by (iv) that $E\|\sum_{i=1}^n X_{ni}\|/a_n \rightarrow 0$, and so (i) holds. Thus the proof of Theorem 2.2 is completed. $\square$

The following theorem states that Theorem 2.2 holds even if the condition (2.5) is replaced by

$$\sum_{n=1}^{\infty} \left(\sum_{i=1}^n \frac{E\|X_{ni}\|^r}{a_n^r} \right)^s < \infty, \quad (2.17)$$

for some $1 \leq r \leq 2$ and $s > 0$.

THEOREM 2.3. *Let $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of rowwise independent B -valued random elements and $\{a_n, n \geq 1\}$ constants such that $0 < a_n \uparrow \infty$. Assume that (2.4) and (2.17) hold. Then the following statements are equivalent.*

- (i) $(1/a_n) \sum_{i=1}^n X_{ni} \rightarrow 0$ in L^1 .
- (ii) $(1/a_n) \sum_{i=1}^n X_{ni} \rightarrow 0$ completely.
- (iii) $(1/a_n) \sum_{i=1}^n X_{ni} \rightarrow 0$ almost surely.
- (iv) $(1/a_n) \sum_{i=1}^n X_{ni} \rightarrow 0$ in probability.

PROOF. Let $\{Y_{ni}\}$ and $\{Z_{ni}\}$ be as in the proof of Theorem 2.2. From the proof of (i) $\Rightarrow$ (ii) in Theorem 2.2, we have

$$\sum_{n=1}^{\infty} \frac{1}{a_n} E \left\| \sum_{i=1}^n Z_{ni} \right\| < \infty, \quad (2.18)$$

which implies $\sum_{i=1}^n Z_{ni}/a_n \rightarrow 0$ in L^1 , completely, almost surely, and in probability. Hence, it is enough to show that

$$\begin{aligned} \frac{1}{a_n} \sum_{i=1}^n Y_{ni} \rightarrow 0 \text{ in } L^1 &\Leftrightarrow \frac{1}{a_n} \sum_{i=1}^n Y_{ni} \rightarrow 0 \text{ completely} \\ &\Leftrightarrow \frac{1}{a_n} \sum_{i=1}^n Y_{ni} \rightarrow 0 \text{ almost surely} \\ &\Leftrightarrow \frac{1}{a_n} \sum_{i=1}^n Y_{ni} \rightarrow 0 \text{ in probability.} \end{aligned} \quad (2.19)$$

Since $Y_{ni} = X_{ni}I(\|X_{ni}\| \leq a_n)$, it follows that $E\psi(\|Y_{ni}\|) \leq E\psi(\|X_{ni}\|)$ and

$$\sum_{n=1}^{\infty} \left(\sum_{i=1}^n \frac{E\|Y_{ni}\|^2}{a_n^2} \right)^s \leq \sum_{n=1}^{\infty} \left(\sum_{i=1}^n \frac{E\|Y_{ni}\|^r}{a_n^r} \right)^s \leq \sum_{n=1}^{\infty} \left(\sum_{i=1}^n \frac{E\|X_{ni}\|^r}{a_n^r} \right)^s. \quad (2.20)$$

Thus $\{Y_{ni}\}$ satisfies the conditions of Theorem 2.2, and so (2.19) holds by Theorem 2.2. $\square$

COROLLARY 2.4. *Let $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of rowwise independent B -valued random elements and $\{a_n, n \geq 1\}$ constants such that $0 < a_n \uparrow \infty$. Assume that $EX_{ni} = 0$ and B is of type r ($1 \leq r \leq 2$). Then (2.4) and (2.17) imply that*

$$\frac{1}{a_n} \sum_{i=1}^n X_{ni} \rightarrow 0 \text{ almost surely.} \quad (2.21)$$

PROOF. By Theorem 2.3, it is enough to show that

$$\frac{1}{a_n} \sum_{i=1}^n X_{ni} \rightarrow 0 \text{ in } L^1. \quad (2.22)$$

Since B is of type r and $EX_{ni} = 0$, it follows by (2.17) that

$$E \left\| \frac{1}{a_n} \sum_{i=1}^n X_{ni} \right\|^r \leq \frac{C_r}{a_n^r} \sum_{i=1}^n E \|X_{ni}\|^r \rightarrow 0, \quad (2.23)$$

and so (2.22) holds. $\square$

REMARK 2.5. The condition (2.3) is weaker than (1.6). Hu and Chung [5] proved Corollary 2.4 under the stronger condition (1.6).

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